Q1: Straightforward induction.

Q2: The Master Theorem cannot be used, because of the logarithm. It is obvious that

\[ T(n) = T(n/2) + n \log n \geq n \log n = \Omega(n \log n). \]

For the upper bound, unfolding gives

\[
T(n) = \sum_{i=0}^{\log n} \frac{n}{2^i} \log \left( \frac{n}{2^i} \right)
\leq \sum_{i=0}^{\log n} \frac{n}{2^i} \log n
= (n \log n) \sum_{i=0}^{\log n} \frac{1}{2^i}
\leq (n \log n) \sum_{i=0}^{\infty} \frac{1}{2^i}
= 2n \log n
= O(n \log n).
\]

Q3: Note that \(1/17 + 16/17 = 1\). Draw the recursion tree and look at the path from the root to the leftmost leaf (always go to the \((1/17)\)-child). This path has \(\Theta(\log n)\) nodes. All levels from the root to this leftmost leaf are full. Each of them contributes \(n\) to \(T(n)\).

Q4: For a quick (and informal) argument: On average, the value of \(k\) is equal to \(n/2\). This gives the recurrence

\[ P(n) = P(n/2) + n^2. \]

Now use the Master Theorem.

This can be done in a formal way, as we did in class for randomized selection.

Q5: For \(i = 1, 2, \ldots, 27\), let \(x_i\) be the \((i \cdot n/27)\)-th smallest number in the sequence. If there is a number that occurs at least \(n/27\) times, then it must be one of these.

Q6: Since both \(x\) and \(y\) can be computed in \(O(n)\) time, we obtain the recurrence

\[ T(n) = 3 \cdot T(n/3) + n. \]

Now use the Master Theorem.

Q7: Draw a heap with 15 nodes, number these nodes from 1 to 15. If you stare at your picture for one minute, you know the answer.
Q8: A max-heap is not a binary search tree.

Q9: In class, we have seen that the time to insert a number into a max-heap is equal to $O(1)$ plus the time for increasekey.

Q10: Use the Master Theorem.

Q11: There is a cycle with 7 vertices.

Q12: If a connected component has $a$ vertices and $b$ edges, then $b \geq a - 1$, because a connected component is connected. This component has a cycle if and only if $b \geq a$.

Q13: See separate file.

Q14: Take the graph with vertices

$$s, a_1, b_1, a_2, b_2, \ldots, a_{(n-2)/2}, b_{(n-2)/2}, t.$$

$s$ has edges to $a_1$ and $b_1$. Each of $a_{(n-2)/2}$ and $b_{(n-2)/2}$ has an edge to $t$. Each $a_i$ has edges to $a_{i+1}$ and $b_{i+1}$. Each $b_i$ has edges to $a_{i+1}$ and $b_{i+1}$.

Q15: This was done in the tutorial.

Q16: If the DFS-forest consists of one tree, then there are no cross edges. Still, there can be a directed cycle (that contains a back edge).

Q17: Take the graph with vertices $A$, $B$, and $C$, and edges $\{A, B\}$ of weight 1, $\{B, C\}$ of weight 1, and $\{A, C\}$ of weight 3.