Midterm COMP 3804

March 1, 2023

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking scheme: Each of the 17 questions is worth 1 mark.

Some useful facts:

1. for any real number $x > 0$, $x = 2^{\log x}$.

2. For any real number $x \neq 1$ and any integer $k \geq 1$,

$$1 + x + x^2 + \cdots + x^{k-1} = \frac{x^k - 1}{x - 1}.$$

3. For any real number $0 < \alpha < 1$,

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1 - \alpha}.$$

Master Theorem:

1. Let $a \geq 1$, $b > 1$, $d \geq 0$, and

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ a \cdot T(n/b) + O\left(n^d\right) & \text{if } n \geq 2. \end{cases}$$

2. If $d > \log_b a$, then $T(n) = O(n^d)$.

3. If $d = \log_b a$, then $T(n) = O(n^d \log n)$.

4. If $d < \log_b a$, then $T(n) = O(n^{\log_b a})$.

1. The Fibonacci numbers are defined by the recurrence

$$\begin{aligned} F_0 &= 0, \\ F_1 &= 1, \\ F_n &= F_{n-1} + F_{n-2} \text{ if } n \geq 2. \end{aligned}$$

   Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

   We define A^0 to be the identity matrix, i.e.,

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   For $n \geq 1$, A^n denotes the matrix

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}.$$

   Is the following true or false? For every integer $n \geq 1$,

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   (a) True.

   (b) False.

2. Consider the recurrence

$$T(n) = T(n/2) + n \log n.$$

   Which of the following is true?

   (a) $T(n) = \Theta(n)$.

   (b) $T(n) = \Theta(n \log n)$.

   (c) $T(n) = \Theta(n \log^2 n)$.

   (d) None of the above.

3. Consider the recurrence

$$T(n) = n + T(n/17) + T(16n/17).$$

Which of the following is true?

(a) $T(n) = \Theta(n)$.

(b) $T(n) = \Theta(n \log n)$.

(c) $T(n) = \Theta(n \log^2 n)$.

(d) None of the above.

4. Consider the following randomized algorithm that takes as input an integer $n \geq 1$:

> **Algorithm** RANDOM(n):
> **if** $n = 1$
> **then** drink one pint of beer
> **else** drink n^2 pints of beer;
>   let k be a uniformly random element in $\{1, 2, \ldots, n\}$;
>   RANDOM(k)
> **endif**

What is the expected number of pints of beer that you drink when you run algorithm RANDOM(n)?

(a) $\Theta(n^3)$

(b) $\Theta(n^2 \log n)$

(c) $\Theta(n^2)$

(d) $\Theta(n)$

5. Consider a sequence of n numbers, where n is a large integer. What is the running time of the fastest comparison-based algorithm that decides if there is a number that occurs at least $n/27$ times in this sequence?

(a) $\Theta(\log n)$

(b) $\Theta(\sqrt{n} \log n)$

(c) $\Theta(n \log n)$

(d) $\Theta(n)$

6. Consider the following variant of QuickSort: Given a sequence of n numbers, compute the $(n/3)$-th smallest element, say x, and the $(2n/3)$-th smallest element, say y. Recursively run the algorithm on all numbers less than x, then recursively run the algorithm on all numbers between x and y, and finally, recursively run the algorithm on all numbers larger than y. What is the running time of this sorting algorithm?

   (a) $\Theta(n)$
   (b) $\Theta(n \log n)$
   (c) $\Theta(n^2)$
   (d) None of the above.

7. Consider a max-heap $A[1 \dots n]$ where $n \geq 15$. Assume that all numbers stored in this max-heap are pairwise distinct. Let x be the fourth largest number stored in this heap. What is the set of indices i such that x may be stored at $A[i]$?

   (a) $\{8, 9, \dots, 15\}$
   (b) $\{4, 5, \dots, 15\}$
   (c) $\{2, 3, \dots, 15\}$
   (d) $\{1, 2, \dots, 15\}$

8. Consider a max-heap $A[1 \dots n]$ where n is a large integer. Assume that all numbers stored in this max-heap are pairwise distinct. How much time does it take to search for an arbitrary number x in this heap?

   (a) $\Theta(1)$
   (b) $\Theta(\log n)$
   (c) $\Theta(n)$
   (d) $\Theta(n \log n)$

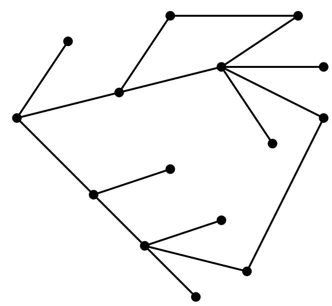
9. Consider a max-heap that stores n pairwise distinct numbers. Professor Uriah Heep has developed a new algorithm that supports the operation IncreaseKey in $O(1)$ time. Using this new algorithm, how much time does it take to insert a number into the max-heap?

   (a) $\Theta(1)$
   (b) $\Theta(h)$, where h is the height of the node storing the new number in the tree visualizing the heap.
   (c) $\Theta(\log n)$
   (d) $\Theta(n)$

10. You are given two recursive algorithms:
Algorithm A solves a problem of size n by recursively solving 3 subproblems, each of size $n/3$, and performing $\Theta(n^3)$ extra time.
Algorithm B solves a problem of size n by recursively solving 125 subproblems, each of size $n/5$, and performing $\Theta(n^2)$ extra time.
Which of these two algorithms is asymptotically faster?

   (a) Algorithm A
   (b) Algorithm B
   (c) Both algorithms have the same running time (up to a constant factor).
   (d) None of the above.

11. Is the following graph bipartite?



   (a) The graph is bipartite.
   (b) The graph is not bipartite.

12. Let $G = (V, E)$ be an undirected graph. An undirected cycle is a sequence $u_1, u_2, \ldots, u_k$ of pairwise distinct vertices, where $k \geq 3$, such that each of $\{u_1, u_2\}$, $\{u_2, u_3\}$, $\ldots$, $\{u_{k-1}, u_k\}$, $\{u_k, u_1\}$ is an edge in E.
Assume the following is given to you: For each connected component of G, you know the number of vertices in this component, and you know the number of edges in this component. Based on this information only, can you decide if G contains an undirected cycle?

   (a) We can decide if G contains an undirected cycle.
   (b) We cannot decide if G contains an undirected cycle.

13. Let $G = (V, E)$ be a directed graph with $V = \{1, 2, \ldots, n\}$. The adjacency matrix of G is an $n \times n$ binary matrix A, where $A_{ij} = 1$ if and only if the directed edge (i, j) is in E.
Assume you are given the adjacency matrix of G. Is it possible to decide, in $O(n)$ time, if there is a vertex with indegree $n - 1$ and outdegree 0?

   (a) This is not possible.
   (b) This is possible.

14. Let $G = (V, E)$ be a directed acyclic graph, let $n = |V|$, and let s and t be two distinct vertices of V. Let $N(s,t)$ denote the number of directed paths in G from s to t.
What is the largest possible value of $N(s,t)$?

    (a) $\Theta(n)$
    (b) $\Theta(n \log n)$
    (c) $\Theta(n^2)$
    (d) This number can be exponential in n.

15. Let $G = (V, E)$ be a directed graph that is given using adjacency lists: Each vertex u has a list $\textsc{Out}(u)$ storing all edges (u, v) going out of u.
What is the running time of the fastest algorithm that computes, for each vertex v, a list $\textsc{In}(v)$ of all edges (u, v) going into v?

    (a) $\Theta((|V| + |E|) \log |V|)$.
    (b) $\Theta(|V| \log |V| + |E|)$.
    (c) $\Theta(|V| + |E| \log |E|)$.
    (d) $\Theta(|V| + |E|)$.

16. Let $G = (V, E)$ be a directed graph. We run depth-first search on G, i.e, algorithm DFS(G).
Is the following true or false?
The graph G has a directed cycle if and only if the DFS-forest has a cross edge.

    (a) True.
    (b) False.

17. Let $G = (V, E)$ be a directed acyclic graph and, for each edge (u, v) in E, let $\textsc{wt}(u, v)$ denote its positive weight. For any two vertices x and y of V, we define $\delta(x, y)$ to be the weight of a shortest path in G from x to y.
We define a new graph $G' = (V, E)$ with the same vertex and edge sets as G. For each edge (u, v) in E, we define its weight in G' to be $\textsc{wt}'(u, v) = \textsc{wt}(u, v) + 7$. For any two vertices x and y of V, we define $\delta'(x, y)$ to be the weight of a shortest path in G' from x to y.
Let x and y be two vertices of V and assume that the shortest path in G from x to y has exactly ℓ edges. Is the following true or false?

$$\delta'(x, y) = \delta(x, y) + 7\ell.$$

    (a) This is always true.
    (b) This is, in general, false.