COMP 3804 — Winter 2026
Solutions Problem Set 6

Question 1: The *set cover problem* is defined as follows:

$$\text{SETCOVER} = \{(B, S_1, S_2, \ldots, S_m, K) : \begin{array}{l} B \text{ is a finite set; } m \text{ is an integer;} \\ S_1, S_2 \ldots, S_m \text{ are sets with } \cup_{i=1}^m S_i = B; \\ K \text{ is an integer;} \\ \text{there exists a subset } I \subseteq \{1, 2, \ldots, m\} \text{ of} \\ \text{size } K, \text{ such that } \cup_{i \in I} S_i = B \ \}. \end{array}$$

Prove that the decision problem SETCOVER is in **NP**.

Solution: The *verification algorithm* $\mathcal{V}$ does the following:

- It takes as input
  - a finite set B, a sequence $S_1, S_2 \ldots, S_m$ of sets such that $\cup_{i=1}^m S_i = B$, and an integer K,
  - and a set I of integers.
- The verification algorithm does the following:
  - Check that $I \subseteq \{1, 2, \ldots, m\}$.
  - Check that the size of I is equal to K.
  - Check that $\cup_{i \in I} S_i = B$.
  - If all of these are correct, then it returns YES. Otherwise, it returns NO.

The *certificate* is of course the set I:

$$\begin{aligned} (B, S_1, \ldots, S_m, K) \in \text{SETCOVER} &\Leftrightarrow \text{there exists } I \subseteq \{1, \ldots, m\} \\ & \text{ such that } |I| = K \text{ and } \cup_{i \in I} S_i = B \\ &\Leftrightarrow \text{there exists a certificate } I \text{ such that} \\ & \ \mathcal{V}(B, S_1, \ldots, S_m, K, I) \text{ returns YES.} \end{aligned}$$

The length of $(B, S_1, \ldots, S_m, K)$ is proportional to

$$\sum_{i=1}^{m} (1 + |S_i|).$$

Note that if S_i is empty, then $1 + |S_i| = 1$; in this case, the input has to specify something like "here is an empty set", which adds a constant to the length of $(B, S_1, \ldots, S_m, K)$.

The length of the certificate I is at most equal to m, which is polynomial in the length of $(B, S_1, \ldots, S_m, K)$, because the length of $(B, S_1, \ldots, S_m, K)$ is at least m.

What is the running time of the verification algorithm:

- Checking that $I \subseteq \{1, 2, \ldots, m\}$ can be done in $O(|I|) = O(m)$ time.
- Checking that $|I| = K$ can be done in $O(|I|) = O(m)$ time.
- Checking that $\cup_{i \in I} S_i = B$ can be done, using brute force, in time

  $$O\left(\sum_{i=1}^{m} (1 + |S_i|) \cdot |B|\right),$$

  which is

  $$O\left(\left(\sum_{i=1}^{m} |S_i| + |B|\right)^2\right),$$

  which is polynomial in the length of $(B, S_1, \ldots, S_m, K)$. (Of course, there are much faster algorithms using sorting and hash tables and balanced binary search trees, etc.)

This shows that SetCover $\in$ **NP**.

Question 2: *Los Tabernacos* is a famous poutine restaurant in Playa del Carmen, Mexico. The owners want to advertize their restaurant to all people ("users") on Instagram. For a given integer K, they ask K users to post a picture of the restaurant[1] on their account.

All users follow the Instagram etiquette: If user u posts a picture, then all users who follow u post a copy of this picture.

Can the owners of Los Tabernacos choose K users such that all Instagram users post a picture of the restaurant?

- Formulate this problem as a decision problem LosTabernacos on a graph.
- Prove that LosTabernacos $\leq_P$ SetCover, i.e., in polynomial time, LosTabernacos can be reduced to SetCover.

Solution: We define a directed graph $G = (V, E)$, where V is the set of all Instagram users. There is an edge (u, v) from u to v, if v follows u. For each vertex u, let $R(u)$ be the set of all vertices v such that there is a directed path from u to v. Note that $u \in R(u)$. Based on this, we get

$$\text{LosTabernacos} = \{(G, K) : \begin{array}{l} G = (V, E) \text{ is a directed graph}, \\ \exists I \subseteq V \text{ such that } |I| = K \text{ and } \cup_{u \in I} R(u) = V \end{array}\}.$$

Next we prove that LosTabernacos $\leq_P$ SetCover. Consider an input (G, K) for LosTabernacos. We map (G, K) to an input

$$f(G, K) = (B, S_1, S_2, \ldots, S_m, K)$$

for SetCover:

[1] and offer them free poutine

- Define $B = V$.
- Define $n = |V|$.
- Define $m = n$.
- Number the vertices of V as $u_1, u_2, \ldots, u_n$.
- For each $i = 1, 2, \ldots, n$, define $S_i = R(u_i)$.
- The value of K remains the same.

By construction, $(G, K) \in$ LosTabernacos is equivalent to $(B, S_1, S_2, \ldots, S_m, K) \in$ SetCover.

How much time is needed to compute $f(G, K)$: To compute $R(u_i)$, we do the following:

- For each vertex v, set $\mathit{visited}(v) = \mathit{false}$.
- Run algorithm Explore(u_i).
- Set $R(u_i)$ to the set of all vertices v for which $\mathit{visited}(v) = \mathit{true}$.

This takes $O(|V|+|E|)$ time per vertex. In total, this takes $O(|V|^2+|V|\cdot|E|) = O((|V|+|E|)^2)$ time, which is polynomial in the length of G.