Carleton University
COMP/MATH 3804 A/B, Test 1

January 28, 2026

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking scheme: Each question is worth 1 mark.

Some useful facts:

1. $1 + 2 + 3 + \cdots + n = n(n+1)/2$.
2. for any real number $x > 0$, $x = 2^{\log x}$.
3. For any real number $x \neq 1$ and any integer $k \geq 1$,
$$1 + x + x^2 + \cdots + x^{k-1} = \frac{x^k - 1}{x - 1}.$$
4. For any real number $0 < \alpha < 1$,
$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1 - \alpha}.$$
5. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \Theta(\log n)$.

Master Theorem:

1. Let $a \geq 1$, $b > 1$, $d \geq 0$, and
$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ a \cdot T(n/b) + \Theta(n^d) & \text{if } n \geq 2. \end{cases}$$
2. If $d > \log_b a$, then $T(n) = \Theta(n^d)$.
3. If $d = \log_b a$, then $T(n) = \Theta(n^d \log n)$.
4. If $d < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.

1. Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of all positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = O(\log n)$. Is it true that there exists a positive constant c such that

$$2^{f(n)} = O\left(n^c\right)?$$

   (a) This is true.

   (b) This is not true.

2. Consider the recurrence

$$T(n) = n^{1/3} + 2 \cdot T(n/4).$$

   Which of the following is true?

   (a) $T(n) = \Theta(n^{1/4})$.

   (b) $T(n) = \Theta(n^{1/3})$.

   (c) $T(n) = \Theta(\sqrt{n})$.

   (d) $T(n) = \Theta(n)$.

3. You are given two algorithms that solve the same problem:

   - Algorithm A solves a problem of size n by dividing it, in $\Theta(n)$ time, into 5 subproblems each of size $n/2$, recursively solving each subproblem, and then combining the solutions in $\Theta(n)$ time.

   - Algorithm B solves a problem of size n by dividing it, in $\Theta(n)$ time, into 2 subproblems each of size $n-1$, recursively solving each subproblem, and then combining the solutions in $\Theta(1)$ time.

   Which of the following is correct?

   (a) The asymptotic running times of algorithms A and B are the same.

   (b) Algorithm A is asymptotically faster than algorithm B.

   (c) Algorithm B is asymptotically faster than algorithm A.

   (d) None of the above.

4. Katy Perry and Justin Trudeau have discovered a recursive algorithm for multiplying two square matrices: Let X and Y be two $n \times n$ matrices, where n is a power of four. If $n \geq 4$, the Perry–Trudeau algorithm computes the product XY in the following way:

   - Eight times, recursively multiply two $\frac{n}{4} \times \frac{n}{4}$ matrices.
   - In $O(n)$ time, combine the solutions to the eight recursive calls, resulting in the product XY.

   Which of the following is true?

   (a) The Perry–Trudeau algorithm may be correct.
   (b) The Perry–Trudeau algorithm is definitely wrong.

5. Let S be a sequence of n distinct numbers, where n is a large integer that is a multiple of 14.

   - Divide S into $n/7$ subsequences, each of length 7.
   - For $i = 1, 2, \ldots, n/7$, let m_i be the median of the i-th subsequence.
   - Let p be the median of $m_1, m_2, \ldots, m_{n/7}$.

   What is the best upper bound on the number of elements in S that are strictly smaller than p?

   (a) $(2/7) \cdot n$
   (b) $(3/7) \cdot n$
   (c) $(4/7) \cdot n$
   (d) $(5/7) \cdot n$

6. Let S be a sequence of n distinct numbers, and let k be an integer with $1 \leq k \leq n$. What is the running time of the fastest comparison-based algorithm that returns the k smallest numbers in S, in sorted order?

   (a) $\Theta(n)$
   (b) $\Theta(n + k \log k)$
   (c) $\Theta(k \log k)$
   (d) None of the above.

7. Let n be a large integer, and let m be an integer with $1 \leq m \leq n$. You are given m lists $A_1, A_2, \ldots, A_m$, each containing n numbers; these lists are not sorted. Consider the following algorithm that sorts the union of these lists into one single sorted list B of length mn:

- $B = \emptyset$.
- For $i = 1, 2, \ldots, m$:
  - $B = B \cup A_i$.
  - $B = \textsc{MergeSort}(B)$.
- Return B.

What is the running time of this algorithm?

(a) $\Theta(mn \log n)$
(b) $\Theta(m^2 n \log n)$
(c) $\Theta(mn^2 \log n)$
(d) $\Theta(m^2 n^2 \log n)$

8. Let n be a large integer, and let m be an integer with $1 \le m \le n$. You are given m lists $A_1, A_2, \ldots, A_m$, each containing n numbers; these lists are not sorted. Assume that m is a power of two. Consider the following algorithm SortManyArrays that sorts the union of these lists into one single sorted list Z of length mn:

   Base case: If $m = 1$:

   - $Z = \text{MergeSort}(A_1)$.
   - Return Z.

   Non-base case: If $m \ge 2$:

   - $X = \text{SortManyArrays}(A_1, A_2, \ldots, A_{m/2})$.
   - $Y = \text{SortManyArrays}(A_{1+m/2}, A_{2+m/2}, \ldots, A_m)$.
   - $Z = \text{Merge}(X, Y)$. (Note: this is the Merge-algorithm from class.)
   - Return Z.

   Let $T(m,n)$ denote the running time of this algorithm. Which of the following is correct?

   (a) $T(m,n) = \Theta(m^2 n \log n)$.
   (b) $T(m,n) = \Theta(m^2 n \log m)$.
   (c) $T(m,n) = \Theta(mn \log m)$.
   (d) $T(m,n) = \Theta(mn \log n)$.

9. Let $A[0,1,2,\ldots]$ be an infinite array. You are given that there exists an integer $n \ge 1$, such that

   $$A[i] = 0 \text{ for all } i = 0, 1, 2, \ldots, n$$

   and

   $$A[i] = 1 \text{ for all } i \ge n+1.$$

   What is the running time of the fastest algorithm that determines the value of n?

   (a) $\Theta(1)$
   (b) $\Theta(n)$
   (c) $\Theta(\log n)$
   (d) $\Theta(n \log n)$

10. After this test, you go to a Karaoke Bar and sing the following randomized and recursive song AWESOMEST(n), which takes as input an integer $n \geq 1$:

> **Algorithm** AWESOMEST(n):
> sing the following line n times:
> *COMP 3804 is the awesomest course I have ever taken*;
> **if** $n \geq 2$
> **then** let k be a uniformly random element in $\{1, 2, \ldots, n\}$;
>   AWESOMEST(k)
> **endif**

What is the expected number of times you sing *COMP 3804 is the awesomest course I have ever taken*?

(a) $\Theta(1)$

(b) $\Theta(\log n)$.

(c) $\Theta(n)$

(d) $\Theta(n \log n)$