Carleton University
COMP/MATH 3804 A/B, Test 3

March 11, 2026

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking scheme: Each question is worth 1 mark.

Some useful facts:

1. $1 + 2 + 3 + \cdots + n = n(n+1)/2$.
2. for any real number $x > 0$, $x = 2^{\log x}$.
3. For any real number $x \neq 1$ and any integer $k \geq 1$,
$$1 + x + x^2 + \cdots + x^{k-1} = \frac{x^k - 1}{x - 1}.$$
4. For any real number $0 < \alpha < 1$,
$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1 - \alpha}.$$
5. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \Theta(\log n)$.

Master Theorem:

1. Let $a \geq 1$, $b > 1$, $d \geq 0$, and
$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ a \cdot T(n/b) + \Theta(n^d) & \text{if } n \geq 2. \end{cases}$$
2. If $d > \log_b a$, then $T(n) = \Theta(n^d)$.
3. If $d = \log_b a$, then $T(n) = \Theta(n^d \log n)$.
4. If $d < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.

1. Let $G = (V, E)$ be a directed acyclic graph and, for each edge (u, v) in E, let $\textsc{wt}(u, v)$ denote its positive weight. Let s and t be two vertices in V.
   Professor Justin Bieber has designed an algorithm that computes the shortest path from s to t, by enumerating all paths from s to t, and then taking the shortest one.
   What is the worst-case running time of Professor Bieber's algorithm?

   (a) $\Theta(|V| + |E|)$
   (b) $\Theta((|V| + |E|) \log |V|)$
   (c) It is at least exponential in the size of V.
   (d) None of the above.

2. Let $G = (V, E)$ be a directed acyclic graph, and let s and t be two distinct vertices of V.
   What is the running time of the fastest algorithm that computes the number of directed paths in G from s to t?

   (a) $\Theta(|V| \cdot |E|)$.
   (b) $\Theta((|V| + |E|) \log |V|)$.
   (c) $\Theta(|V| + |E|)$.
   (d) $\Theta(|E|)$.

3. Let $G = (V, E)$ be a directed graph. We run depth-first search on G, i.e., algorithm DFS(G).
   Is the following true or false?
   There can be an edge (v, u) in G, such that

   $$\textsc{pre}(u) < \textsc{pre}(v) < \textsc{post}(u) < \textsc{post}(v).$$

   (Note that PRE and POST are the pre- and post-numbers that are computed during algorithm DFS(G).)

   (a) True
   (b) False

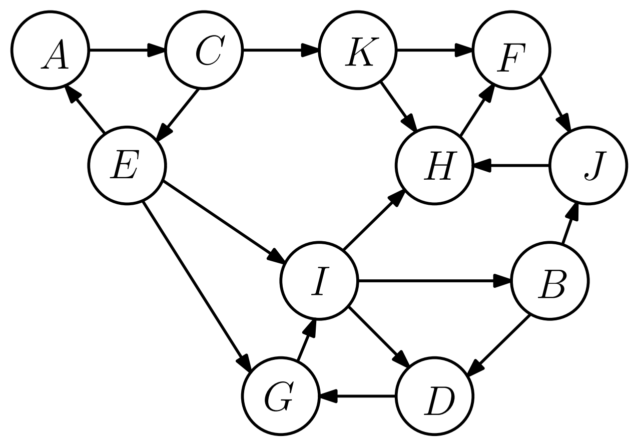
4. Let $G = (V, E)$ be a directed graph. We run depth-first search on G, i.e., algorithm DFS(G).
   Recall that this classifies each edge of E as a tree edge, forward edge, back edge, or cross edge.
   Is the following true or false?
   If G contains a directed cycle that contains a cross edge, then G also contains a directed cycle that contains a back edge.

   (a) True
   (b) False

5. Let $G = (V, E)$ be a directed graph. We run depth-first search on G, i.e, algorithm DFS(G). Recall that this classifies each edge of E as a tree edge, forward edge, back edge, or cross edge.
Let (u, v) be an edge of E that is not classified as a tree edge.
Is the following true or false?
It is possible to run algorithm DFS(G), where vertices and edges are processed in a different order, such that (u, v) is classified as a tree edge.

   (a) True.
   (b) False.

6. Consider the following directed graph:



We run algorithm DFS (depth-first search) on this graph. Whenever there is a choice of vertices, we pick the one that is alphabetically first. Thus, the first call to EXPLORE is for the vertex A.
In what order will the nodes be visited during DFS?

   (a) $A, C, E, G, I, B, D, H, F, J, K$
   (b) $A, C, E, G, I, B, D, J, H, F, K$
   (c) $A, C, E, G, I, D, B, H, F, J, K$
   (d) None of the above.

7. Let G be a directed graph with n vertices and m edges. This graph is represented using the adjacency matrix.
What is the running time of the fastest algorithm that runs DFS (depth-first search) on this graph?

   (a) $\Theta(m)$
   (b) $\Theta(n + m)$
   (c) $\Theta(n)$
   (d) $\Theta(n^2)$

8. Let $G = (V, E)$ be a directed acyclic graph. We run algorithm DFS (depth-first search) on this graph. After this algorithm has terminated, each vertex has a PRE- and POST-number.
Is the following true or false?
For every edge (u, v) in E, $\text{POST}(u) < \text{PRE}(v)$.

   (a) True
   (b) False

9. Let $G = (V, E)$ be a directed graph.
When does G have exactly one topological sorting?

   (a) Whenever G is a directed acyclic graph.
   (b) Whenever G has a unique cycle.
   (c) Whenever G contains a directed path that visits every vertex exactly once.
   (d) None of the above.

10. Let $G = (V, E)$ be a directed graph with n vertices and m edges. This graph is represented using adjacency lists. Assume that each edge (u, v) in E has a positive weight $\textsc{wt}(u, v)$. For a given source vertex s, you want to run Dijkstra's algorithm.
Unfortunately, you only have an implementation of a *max-heap* that supports the following operations:

    - Building a max-heap for n numbers takes $O(n)$ time.
    - The operation ExtractMax on a max-heap of size n takes time $f(n)$.
    - The operation IncreaseKey on a max-heap of size n takes time $g(n)$.

    This implementation is "locked" in the sense that you cannot change it. In other words, this implementation can only be used as a black box.
After lots of thinking, you realize that you can use this max-heap to run Dijkstra's algorithm. What is the running time of Dijkstra's algorithm if you use this max-heap?

    (a) $\Theta((n + m) \cdot (f(n) + g(n)))$
    (b) $\Theta(m \cdot f(n) + n \cdot g(n))$
    (c) $\Theta(n \cdot f(n) + m \cdot g(n))$
    (d) None of the above.

11. Let $G = (V, E)$ be a connected undirected graph with n vertices and m edges. Assume that each edge $\{u, v\}$ in E has a positive weight $\textsc{wt}(u, v)$.
Recall that a *minimum spanning tree* is a spanning tree of G, whose total edge weight is minimum (among all possible spanning trees).
You have seen in class that Kruskal's algorithm computes a minimum spanning tree of G by

    - sorting the edges of E by their weights in $\Theta(m \log m)$ time, and
    - using the Union-Find data structure to process a sequence of $n - 1$ Union-operations and $2m$ Find-operations; this takes $\Theta(m + n \log n)$ time.

    The guy in front of the classroom concluded that the total running time of the algorithm is $\Theta(m \log n)$.
Justin Bieber is doubtful about this. Justin claims that the actual total running time is $\Theta(m \log m + n \log n)$, which is not the same as $\Theta(m \log n)$.
Is Justin's claim correct?

    (a) Justin's claim is correct.
    (b) Justin's claim is not correct.

12. Let $G = (V, E)$ be a directed graph that is given using adjacency lists: Each vertex u has a list $\textsc{Out}(u)$ storing all edges (u, v) going out of u.
What is the running time of the fastest algorithm that computes, for each vertex v, a list $\textsc{In}(v)$ of all edges (u, v) going into v?

    (a) $\Theta(|V| + |E|)$.
    (b) $\Theta(|V| \log |V| + |E|)$.
    (c) $\Theta(|V| + |E| \log |E|)$.
    (d) $\Theta((|V| + |E|) \log |V|)$.

13. Let $G = (V, E)$ be a directed acyclic graph and, for each edge (u, v) in E, let $\textsc{wt}(u, v)$ denote its positive weight. For any two vertices x and y of V, we define $\delta(x, y)$ to be the weight of a shortest path in G from x to y.
We define a new graph $G' = (V, E)$ with the same vertex and edge sets as G. For each edge (u, v) in E, we define its weight in G' to be $\textsc{wt}'(u, v) = \textsc{wt}(u, v) + 7$. For any two vertices x and y of V, we define $\delta'(x, y)$ to be the weight of a shortest path in G' from x to y.
Let x and y be two vertices of V and assume that the shortest path in G from x to y has exactly ℓ edges. Is the following true or false?

$$\delta'(x, y) = \delta(x, y) + 7\ell.$$

    (a) This is always true.
    (b) This is, in general, false.

14. Let $G = (V, E)$ be a connected undirected graph, in which each edge $\{u, v\}$ has a weight $\textsc{wt}(u, v)$. Assume that all edge weights are distinct.
Consider an arbitrary partition of the vertex set V into two disjoint subsets A and B. Let $\{a, b\}$ be the edge in E of minimum weight, with $a \in A$ and $b \in B$.
Is the following statement true or false?

$\{a, b\}$ is the only edge in the minimum spanning tree between the sets A and B.

    (a) The statement is always true.
    (b) The statement is, in general, false.

15. Let $G = (V, E)$ be a connected undirected graph, in which each edge $\{u, v\}$ has a weight $\text{WT}(u, v)$. Consider the following algorithm.

Step 1: Set $E' = E$, $G' = (V, E')$, and $E'' = E$.

Step 2: While E'' is not empty:

Step 2.1: Compute an edge e of largest weight in E'', and set $E'' = E'' \setminus \{e\}$.

Step 2.2: If the graph with vertex set V and edge set $E' \setminus \{e\}$ is connected, then set $E' = E' \setminus \{e\}$ and $G' = (V, E')$.

Step 3: Return the graph $G' = (V, E')$.

Which of the following is correct?

(a) The output is always a minimum spanning tree of the input graph G.

(b) The output is not necessarily a minimum spanning tree of the input graph G.