Carleton University
COMP/MATH 3804 A/B, Test 4

March 25, 2026

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking scheme: Each question is worth 1 mark.

Some useful facts:

1. $1 + 2 + 3 + \cdots + n = n(n+1)/2$.

2. for any real number $x > 0$, $x = 2^{\log x}$.

3. For any real number $x \neq 1$ and any integer $k \geq 1$,

$$1 + x + x^2 + \cdots + x^{k-1} = \frac{x^k - 1}{x - 1}.$$

4. For any real number $0 < \alpha < 1$,

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1 - \alpha}.$$

5. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \Theta(\log n)$.

Master Theorem:

1. Let $a \geq 1$, $b > 1$, $d \geq 0$, and

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ a \cdot T(n/b) + \Theta(n^d) & \text{if } n \geq 2. \end{cases}$$

2. If $d > \log_b a$, then $T(n) = \Theta(n^d)$.

3. If $d = \log_b a$, then $T(n) = \Theta(n^d \log n)$.

4. If $d < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.

1. Let $G = (V, E)$ be a directed graph in which each edge (u, v) has a positive weight $\textsc{wt}(u, v)$, and let s be a vertex in V. For each vertex v in V, and each integer $k \geq 0$, let $\textsc{dist}(v, k)$ be the length of the shortest path from s to v that has k edges.
Which of the following is correct for any vertex $v \neq s$ and any integer $k \geq 1$?

   (a) $\textsc{dist}(v, k) = \min\{\textsc{dist}(u, k-1) + \textsc{wt}(u, v) : (v, u) \text{ is an edge in } E\}$
   (b) $\textsc{dist}(v, k) = \min\{\textsc{dist}(u, k-1) + \textsc{wt}(u, v) : (u, v) \text{ is an edge in } E\}$
   (c) $\textsc{dist}(v, k) = \min\{\textsc{dist}(u, k) + \textsc{wt}(u, v) : (v, u) \text{ is an edge in } E\}$
   (d) $\textsc{dist}(v, k) = \min\{\textsc{dist}(u, k) + \textsc{wt}(u, v) : (u, v) \text{ is an edge in } E\}$

2. Let $G = (V, E)$ be an undirected path graph, where

   - $V = \{u_1, u_2, \ldots, u_n\}$ and
   - $E = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \ldots, \{u_{n-1}, u_n\}\}$.

   Assume that each vertex u_i has a postive weight $\omega(u_i)$.
An *independent set* in G is a subset I of V such that for any two distinct vertices u and v in I, $\{u, v\}$ is not an edge in E.
A *maximum-weight independent set* is an independent set I whose total weight $\sum_{u \in I} \omega(u)$ is maximum.
For $i = 0, 1, \ldots, n$, let G_i be the subgraph of G that only contains the vertices $u_1, u_2, \ldots, u_i$ and the edges $\{u_1, u_2\}, \{u_2, u_3\}, \ldots \{u_{i-1}, u_i\}$, and let W_i be the total weight of a maximum-weight independent set in G_i.
Which of the following is correct for any $i = 2, 3, \ldots, n$?

   (a) $W_i = \max(W_{i-1}, W_{i-2} + \omega(u_i))$
   (b) $W_i = \max(W_{i-2}, W_{i-1} + \omega(u_i))$
   (c) $W_i = \max(W_{i-1}, W_{i-2})$
   (d) $W_i = \max(W_{i-1} + \omega(u_i), W_{i-2} + \omega(u_i))$

3. The *subset sum problem* is defined as follows:

   $$\textsc{SubSetSum} = \{(a_1, a_2, \ldots, a_m, b) : \begin{array}{l} m, a_1, a_2, \ldots, a_m, b \text{ are positive integers and} \\ \exists I \subseteq \{1, 2, \ldots, m\} \text{ such that } \sum_{i \in I} a_i = b \ \}. \end{array}$$

   Is the following true or false?
Any input $(a_1, a_2, \ldots, a_m, b)$ to $\textsc{SubSetSum}$ can be solved in $O(mb)$ time.

   (a) True.
   (b) False.

4. You are given a set $S = \{1, 2, \ldots, n\}$ of n cities, and an $n \times n$ matrix C, where C_{ij} is the (positive) cost to travel from city i to city j. Note that C_{ij} is not necessarily equal to C_{ji}. For any city i in S and any subset A of $S \setminus \{i\}$, let $mc(i, A)$ be the minimum cost of any path that starts in city i and visits all cities of A (such a path ends in a city of A).
Which of the following is correct if $|A| \geq 2$?

   (a) $mc(i, A) = \min_{j \in A} (C_{ji} + mc(i, A \setminus \{j\}))$.
   (b) $mc(i, A) = \min_{j \in A} (C_{ji} + mc(i, A))$.
   (c) $mc(i, A) = \min_{j \in A} (C_{ij} + mc(j, A \setminus \{j\}))$.
   (d) $mc(i, A) = \min_{j \in A} (C_{ij} + mc(j, A))$.

5. In the SubSetSum problem, the input is a tuple $(a_1, a_2, \ldots, a_n, t)$ of positive integers. The question is if there exists a subset I of $\{1, 2, \ldots, n\}$ such that $\sum_{i \in I} a_i = t$.
For each k and ℓ with $1 \leq k \leq n$ and $0 \leq \ell \leq t$, let the Boolean variable $B(k, \ell)$ be True if and only if there exists a subset I of $\{1, 2, \ldots, k\}$ such that $\sum_{i \in I} a_i = \ell$.
For each k and ℓ with $1 \leq k \leq n$ and $\ell < 0$, let the Boolean variable $B(k, \ell)$ be False.

   Which of the following is correct for $2 \leq k \leq n$ and $0 \leq \ell \leq t$?

   (a) $B(k, \ell) =$ True if and only if $B(k-1, \ell) =$ True and $B(k-1, \ell + a_k) =$ True.
   (b) $B(k, \ell) =$ True if and only if $B(k-1, \ell) =$ True and $B(k-1, \ell - a_k) =$ True.
   (c) $B(k, \ell) =$ True if and only if $B(k-1, \ell) =$ True or $B(k-1, \ell + a_k) =$ True.
   (d) $B(k, \ell) =$ True if and only if $B(k-1, \ell) =$ True or $B(k-1, \ell - a_k) =$ True.

6. In the *Longest Increasing Subsequence* problem, we are given a sequence $X = (x_1, x_2, \ldots, x_n)$ of n numbers, and we want to compute the length LIS(X) of a longest increasing subsequence of X.
In the *Longest Common Subsequence* (LCS) problem, we are given two sequences $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_m)$ of numbers, and we want to compute the length LCS(X, Y) of a longest subsequence that occurs in X and Y.
Is the following statement true or false?
Let $X = (x_1, x_2, \ldots, x_n)$ be a sequence of n distinct numbers, and let Y be the permutation of X such that the numbers in Y are in sorted order. Then

$$\text{LIS}(X) = \text{LCS}(X, Y).$$

   (a) This statement is always true.
   (b) This statement is, in general, false.

7. Let $G = (V, E)$ be a directed graph in which each edges (u, v) has a positive weight $\textsc{wt}(u, v)$. Let u and v be two vertices, and let P be the longest path from u to v. Let w be a vertex on this path P.
   Is the following statement true or false?
   The subpath of P from u to w must be the longest path from u to w.

   (a) This statement is always true.

   (b) This statement is, in general, false.

8. Let

$$\textsc{HamiltonCycle} = \{G : \text{ graph } G \text{ has a Hamilton cycle}\}$$

   and

$$\textsc{NonHamiltonCycle} = \{G : \text{ graph } G \text{ does not have a Hamilton cycle}\}.$$

   Is the following statement true or false?
   In class, we have seen a proof that HamiltonCycle is in **NP**. By swapping YES and NO in this proof, we obtain a proof that NonHamiltonCycle is in **NP**.

   (a) The statement is true.

   (b) The statement is false.

9. In the SubSetSum problem, the input is a tuple $(x_1, x_2, \ldots, x_n, t)$ of positive integers.
   In the decision problem (i.e., the answer is YES or NO), we have to decide if there exists a subset I of $\{1, 2, \ldots, n\}$ such that $\sum_{i \in I} x_i = t$.
   Assume we have an algorithm A that solves the decision problem in time $T(n)$. Note that this algorithm only returns YES or NO; it does not return anything else.
   We want to design an algorithm B that, in case the answer to the question is YES, returns a subset I as above.
   Which of the following is correct?

   (a) Any such algorithm B must have a running time of $\Omega(2^{T(n)})$.

   (b) There exists such an algorithm B with running time $O(n^c \cdot T(n))$, for some constant c.

10. What does **NP** stand for?

   (a) Non-polynomial time

   (b) No parking

   (c) Nurse practitioner

   (d) Non-deterministic polynomial time.

11. Consider the following decision problem:
Input: A sequence $X = (x_1, x_2, \ldots, x_n)$ of n numbers, and an integer $\ell \geq 1$.
Question: Is the length of a longest increasing subsequence of X at least ℓ?
Which of the following is correct?

    (a) This decision problem is in **P**.

    (b) It is not known if this decision problem is in **P**.

12. Let L be a decision problem of the following form: Given an input I, does there exist a solution, say J, that has some property, say K?
Assume that for all inputs I of length n, the number of possible solutions J is at least 2^n. Note that not all of these solutions may satisfy property K.
Is the following true or false?
The decision problem L is not in **P**.

    (a) True

    (b) False