First-Passage Percolation Time on Hypercubes

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Abstract

We give a simple proof of the following statement: If one puts independent exponential mean 1 edge weights on the edges of a d-cube, then the expected weight of the lightest path from $(0,\dots,0)$ to $(1,\dots,1)$ is $O(1)$.

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1 Introduction

The d-cube Q_d is the graph with vertex set $V(Q_d) = \{0,1\}^d$ and whose edge set $E(Q_d)$ contains the edge uv if and only if the Hamming distance between u and v is exactly 1. Assign an independent exponential(1) *edge weight* to each edge of Q_d. These weights define a lightest path metric between the vertices of Q_d, where $w(u,v)$ denotes the weight of the lightest path from vertex u to vertex w. What is the expected weight $\mathrm{E}[w(\mathbf{0},\mathbf{1})]$ of the lightest path from $\mathbf{0} = (0,\dots,0)$ to $\mathbf{1} = (1,\dots,1)$?

In this note, we offer a simple proof that $\mathrm{E}[w(\mathbf{0},\mathbf{1})] \in O(1)$; although the number of edges in any path from $\mathbf{0}$ to $\mathbf{1}$ is at least d, the expected weight of the lightest path does not increase with d. This result is not new. Indeed, this type of question is central in the study of *first-passage percolation* as introduced by Hammersley and Welsh in 1965 [6] and recently surveyed by Auffinger *et al* [2].

The question we consider here was first asked by Aldous [1, Section G7] and first answered by Fill and Pemantle [5] who showed that the weight of the lightest monotone path from $\mathbf{0}$ to $\mathbf{1}$ converges in probability to 1 as $d \to \infty$. The most recent result on this problem is due to Martinsson [7], who shows that $w(\mathbf{0},\mathbf{1})$ converges in probability to $\ln(1+\sqrt{2}) \approx 0.881$ as $d \to \infty$. The difference between these two results is that Fill and Pemantle's paths are *monotone*—they have exactly d edges—while Martinsson's paths may have more than d edges.

The proof we present here is (arguably) simpler and more accessible to a computer science audience than either of the proofs discussed above. On the other hand, our proof only gives an $O(1)$ upper bound (approximately 303.61) on the weight of the lightest path from $\mathbf{0}$ to $\mathbf{1}$ and doesn't give any lower bound. It also does not guarantee a monotone path; it produces paths using roughly $3d/2$ edges.

One nice feature of this new proof is that it employs a natural greedy strategy that results in an algorithm for finding an $O(1)$ weight path that runs in $O(d^4)$ time. In distributed computing terms, this algorithm is 3-local, it can be implemented by an agent that only has information about edge weights in a neighbourhood of radius 3 about the current vertex.

2 Review of Probability Concepts

Recall that an exponential(λ) random variable X has a distribution defined by

$$\Pr\{X > x\} = e^{-\lambda x} \ , \quad x \geq 0 \ ,$$

and has expected value

$$\mathrm{E}[X] = \int_0^\infty \Pr\{X > x\}\, dx = \int_0^\infty e^{-\lambda x}\, dx = 1/\lambda$$

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If $X_1, \ldots, X_k$ are independent exponential(1) random variables, then their minimum is an exponential(k) random variable.

$$\Pr\{\min\{X_1, \ldots X_k\} > x\} = \Pr\{X_1 > x,\ X_2 > x, \ldots, \text{ and } X_k > x\} = \left(e^{-x}\right)^k = e^{-kx} \ .$$

At one point, we will make use of a simple Chernoff bound for binomial random variables. If B is a binomial random variable with expected value $\mathrm{E}[B]$, then

$$\Pr\{B < \mathrm{E}[B]/2\} \le e^{-\mathrm{E}[B]/8} \ . \tag{1}$$

3 Some Intuition

Before continuing, we first describe a naïve greedy algorithm that does not quite work. Suppose that, to route from $\mathbf{0}$ to $\mathbf{1}$ we employ the strategy of repeatedly taking the lightest edge that takes us closer to $\mathbf{1}$. At the zeroth step, there are d edges to choose from, so the lightest one will have a weight that is the minimum of d exponential(1) random variables, i.e., it is an exponential(d) random variable and its expected value is $1/d$. At the first step, there are $d-1$ edges to choose from, so the expected weight of the edge we choose is $1/(d-1)$. In general, the expected weight of an edge we choose at the ith step is $1/(d-i)$. Thus, the expected weight of the edges crossed by this greedy algorithm is

$$\sum_{i=0}^{d-1} 1/(d-i) = \sum_{i=1}^{d} 1/i \le \ln d + 1 \ .$$

This is not quite the $O(1)$ bound we are hoping for, but it is significantly better than the obvious $O(d)$ bound.

The problem with this greedy algorithm is that it works well for the first $d/2$ steps, but the cost of each step increases as it gets closer to $\mathbf{1}$, eventually yielding the dth harmonic number. Our solution to this problem is to employ a *foxtrot* in the second $d/2$ steps, in which we repeatedly take a step away from $\mathbf{1}$ followed by two steps toward $\mathbf{1}$. In the ith stage, this allows us to choose from among $i(d-i)^2$ different paths of length three instead of being restricted $d-i$ paths of length 1. Next, we prove a lemma that allows us to analyze these foxtrot steps.

4 Trees of Height 3

The following result, depicted in Figure 1, studies a first-passage percolation problem on a tree of height three.

▶ **Lemma 1.** *Let $a, b, c \ge 1$ be integers and let T be a rooted tree of height three of whose root has a children, each of which has b children, each of which has c children. Assign an* exponential(1) *edge weight to each edge of T and let $\rho(T)$ denote the weight of the lightest root-to-leaf path in T. Then*

$$\Pr\{\rho(T) > t\} \le e^{-at/64} + e^{-bat^2/1024} + e^{-cbat^3/768} \ .$$

Our only use for Lemma 1 is to upper bound the expected value of $\rho(T)$. We do this now, before proving Lemma 1.

▶ **Corollary 2.** *Let a, b, c, T, and $\rho(T)$ be defined as in Lemma 1, with $a \ge b \ge c \ge 1$. Then* $\mathrm{E}[\rho(T)] \le C/(abc)^{1/3}$, *for $C = 64 + 16\sqrt{\pi} + 16(1/12)^{2/3}\Gamma(1/3)$.*

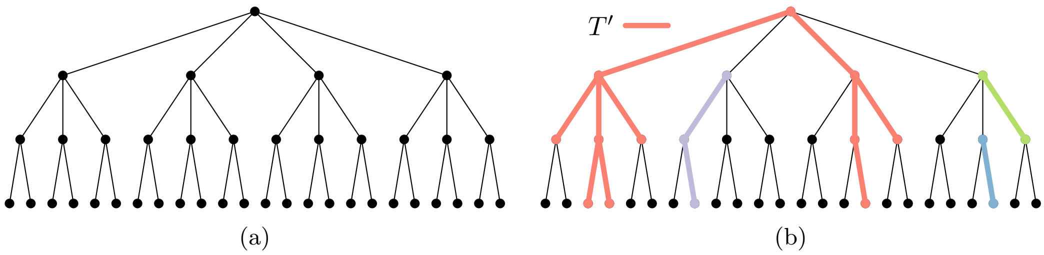

Figure 1 (a) The tree T for $a, b, c = 4, 3, 2$. After removing edges of weight greater than $t/3$, we study the component T' containing the root of T.

Proof. Recall that, for any non-negative random variable X, $\mathrm{E}[X] = \int_0^\infty \Pr\{X > x\}\, dx$. Therefore,

$$\begin{aligned}
\mathrm{E}[\rho(T)] &= \int_0^\infty \Pr\{\rho(T) > t\}\, dt \\
&\le \int_0^\infty \left(e^{-at/64} + e^{-bat^2/1024} + e^{-cbat^3/768}\right) dt \\
&= \frac{64}{a} + \frac{16\sqrt{\pi}}{\sqrt{ab}} + \frac{16(1/12)^{2/3}\Gamma(1/3)}{\sqrt[3]{abc}} \\
&\le \frac{64 + 16\sqrt{\pi} + 16(1/12)^{2/3}\Gamma(1/3)}{\sqrt[3]{abc}}
\end{aligned}$$

where the last inequality uses the assumption that $a \ge b \ge c$. ◀

Proof of Lemma 1. For large values of t, the proof is simple. In particular, if $t \ge 6 \ln 3$, then we observe that T contains a edge-disjoint root-to-leaf paths. For one of these paths to have weight greater than t, at least one of its three edges must have weight greater than $t/3$. The probability that this occurs (for a single path) is at most $3e^{-t/3}$. Since the paths are edge-disjoint, their weights are independent, so the probability that it occurs for all a paths is therefore at most

$$(3e^{-t/3})^a = (e^{\ln 3 - t/3})^a = (e^{(\ln 3/t - 1/3)t})^a \le (e^{-t/6})^a = e^{-at/6} \ ,$$

where the inequality uses the assumption that $t \ge 6 \ln 3$.

We now move on to the interesting case, where $0 \le t < 6 \ln 3$. Imagine removing every edge of T having weight greater than $t/3$ to obtain a forest F and let T' be the tree in F that contains the root of T. For each $i \in \{0, 1, \dots, 3\}$, let N_i denote the number of nodes of T' having depth i. Observe that, if $N_3 \ge 1$, then there is a root-to-leaf path in T of weight at most t. Therefore the rest of the proof is devoted to upper bounding $\Pr\{N_3 = 0\}$.

Observe that N_1 is a binomial$(a, 1 - e^{-t/3})$ random variable. The probability $1 - e^{-t/3}$ is a bit unwieldy so we observe that, in the range $0 \le t < 6 \ln 3$, $1 - e^{-t/3} \ge t/8$. Therefore, we can lower bound the expected value

$$\mathrm{E}[N_1] = a(1 - e^{-t/3}) \ge at/8 \ .$$

Since N_1 is binomial, by (1),

$$\Pr\{N_1 < at/16\} \le e^{-at/64} \ .$$

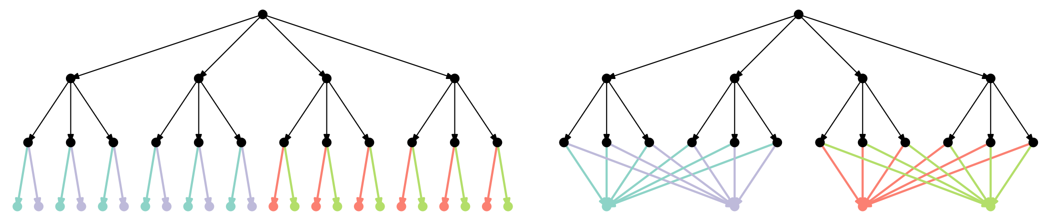
(a) (b)

Figure 2 The result of Lemma 1 holds in a slightly more general setting, in which some of the leaves of T are identified.

Now, conditioned on N_1, N_2 is a binomial$(bN_1, 1-e^{-t/3})$ random variable and

$$\mathrm{E}[N_2 \mid N_1 \geq at/16] \geq bat(1-e^{-t/3})/16 \geq bat^2/128 \enspace .$$

Again, (1) yields

$$\Pr\{N_2 < bat^2/256 \mid N_1 \geq at/16\} \leq e^{-bat^2/1024} \enspace .$$

Now, conditioned on N_2, N_3 is a binomial$(cN_2, 1-e^{-t/3})$ random variable but for this last step we don't need Chernoff's help:

$$\Pr\{N_3 = 0 \mid N_2 \geq bat^2/256\} \leq (e^{-t/3})^{cbat^2/256} = e^{-cbat^3/768} \enspace .$$

Summarizing,

$$\begin{aligned}\Pr\{\rho(T) > t\} &\leq \Pr\{N_3 = 0\} \\ &\leq \Pr\{N_3 = 0 \mid N_2 \geq bat^2/256\} \\ &\quad + \Pr\{N_2 < bat^2/256 \mid N_1 \geq at/16\} \\ &\quad + \Pr\{N_1 < at/16\} \\ &\leq e^{-at/64} + e^{-bat^2/1024} + e^{-cbat^3/768} \enspace .\end{aligned}$$ ◀

▶ Remark. We note that the result of Lemma 1 also holds in a slightly more general setting, an example of which is illustrated in Figure 2. In particular, we can identify groups of leaves of T arbitrarily to obtain a directed acyclic graph D in which the leaves of T become sinks in D. The lemma then gives bounds on the probability that the weight of the lightest root-to-sink path exceeds t.

5 The Proof

▶ **Theorem 3.** *Let Q_d be the d-cube equipped with independent* exponential(1) *edge weights. Then the expected weight of the lightest path from $\mathbf{0} = (0,\ldots,0)$ to $\mathbf{1} = (1,\ldots,1)$ is $O(1)$.*

Proof. For each $i \in \{0,\ldots,d\}$, let L_i denote set of vertices of Q_d whose distance from $\mathbf{0}$ is exactly i, so that $\mathbf{0} \in L_0$ and $\mathbf{1} \in L_d$. We use a greedy strategy to find a path from $\mathbf{0}$ to $\mathbf{1}$. To get from a vertex $u \in L_i$ to some vertex in L_{i+1} the strategy does one of the following two things:

1. If $i < d/2$, then the algorithm traverses the lightest edge joining u to some vertex in L_{i+1}. The weight of this edge is the minimum of $d-i$ independent exponential(1) random variables, so the expected weight of this edge is $1/(d-i) \leq 2/d$.

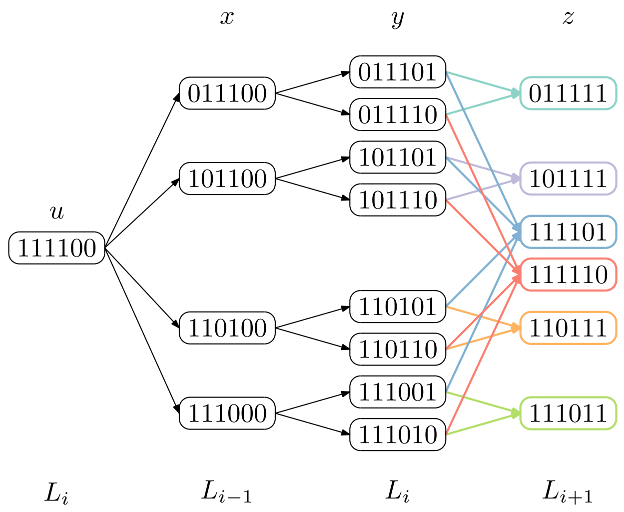

Figure 3 A foxtrot step considers all paths $uxyz$ with $x \in L_{i-1}$, $y \in L_i \setminus \{u\}$ and $z \in L_{i+1}$.

2. If $i \geq d/2$, we consider the $i(d-i)^2$ paths $uxyz$ with $x \in L_{i-1}$, $y \in L_i \setminus \{u\}$, and $z \in L_{i+1}$ and traverse the lightest such path. See Figure 3. This set of paths has the structure of the DAG described in the remark at the end of Section 4: The root has i outgoing edges and the nodes at depth 1 and 2 each have $d-i$ outgoing edges. Therefore, by Corollary 2, the expected weight of the three edges traversed in this step is at most $Ci^{-1/3}(d-i)^{-2/3}$.

Therefore, the expected weight of the entire path found by this algorithm is at most

$$
\begin{aligned}
\mu &\leq \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} 2/d + \sum_{i=\lceil d/2 \rceil}^{d-1} Ci^{-1/3}(d-i)^{-2/3} \\
&\leq 2 + C\left(\sum_{i=\lceil d/2 \rceil}^{d-1} i^{-1/3}(d-i)^{-2/3}\right) \\
&\leq 2 + C(d/2)^{-1/3}\left(\sum_{i=\lceil d/2 \rceil}^{d-1} (d-i)^{-2/3}\right) \\
&= 2 + C(d/2)^{-1/3}\left(\sum_{k=1}^{\lfloor d/2 \rfloor} k^{-2/3}\right) \\
&\leq 2 + C(d/2)^{-1/3}\left(1 + \int_1^{d/2} x^{-2/3}\,dx\right) \\
&= 2 + C(d/2)^{-1/3}\left(3(d/2)^{1/3} - 3\right) \\
&\leq 2 + 3C \ .
\end{aligned}
$$
◀

6 Discussion

Our proof works for any edge weight distribution with a probability density function that is strictly positive in some right neighbourhood of 0 and whose tail decays exponentially. These properties ensure the minimum of k independent random samples from the distribution has expected value $O(1/k)$. The second property also ensures that there is are constants $T, c > 0$ such that, for all $t > T$, $\Pr\{X > t\} \le e^{-ct}$. Besides the exponential(λ) distribution for constant λ, another notable example is the uniform distribution over the interval $[0, 1]$.

The weight of the path found in our proof is the sum of d random variables. The first $d/2$ of these are independent exponential(1). The second $d/2$ can be split into two subsets (the odd steps and the even steps) that are each independent. Using standard methods for deriving concentration inqualities along with the fact that Lemma 1 gives an exponential tail bound on $\rho(T)$, it is possible to show that, for any $\delta > 0$, $\Pr\{\mu \ge (1+\delta)(1+3C)\} \to 0$ as $d \to \infty$. Unfortunately, the rate of this convergence is not quite enough to prove that with high probability there is a path of weight $O(1)$ from $\mathbf{0}$ to *every* vertex of Q_d. This latter fact is something that Fill and Pemantle's proof does manage to show [5].

A conjecture of Aldous [1, Conjecture G7.1] was the original motivation for the work of Fill and Pemantle. In his discussion of this conjecture, Aldous describes the naïve greedy algorithm from Section 3 and shows that it produces a path whose expected weight is the dth harmonic number. In their review of previous work, Fill and Pemantle [5] point out that similar results on percolation were proven for complete binary trees [8]. 24 years later, our proof shows that the result for hypercubes is a consequence of the naïve greedy algorithm and a result for trees of height three.

The proof of Fill and Pemantle [5] and an unpublished proof of Bollobás *et al* [3] both work by a careful analysis of the $d!$ monotone paths from $\mathbf{0}$ to $\mathbf{1}$, and the ways in which pairs of these paths can overlap. Fill and Pemantle require this because they use (a variant of) the second moment method, while Bollobás *et al* use a lemma due to Janson that also has conditions on the interactions between pairs of objects. Our proof sidesteps all of this.

Martinsson's proof [7] works by relating this problem to a so-called *branching translation process* and then using a variety of advanced probabilistic tools to study this process. This branching translation process was used by Fill and Pemantle's original work to provide a lower bound on $w(\mathbf{0}, \mathbf{1})$.

Finally, we point out that the first-passage percolation time in a graph G with i.i.d. exponential edge weights is closely related to the the maximum number of edges, $h(G, s)$, in the lightest path from some vertex s any other vertex of G. Devroye *et al* [4] describe a relationship between first-passage percolation time, the number of simple paths in G and $h(G, s)$ and use this to derive bounds on $\mathrm{E}[h(G, s)]$ that are tight for many classes of graphs.

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