Testing the Quality of Manufactured Disks and Cylinders*

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Abstract. We consider the problem of testing the roundness of a manufactured object using the finger probing model of Cole and Yap [1]. When the object being tested is a disk and it's center is known, we describe a procedure which uses $O(n)$ probes and $O(n)$ computation time. (Here $n = |1/q|$, where q is the quality of the object.) When the center of the object is not known, a procedure using $O(n)$ probes and $O(n \log n)$ computation time is described. When the object being tested is a cylinder of length l, a procedure is described which uses $O(ln^2)$ probes and $O(ln^2 \log ln)$ computation time. Lower bounds are also given which show that these procedures are optimal in terms of the number of probes used.

1 Introduction

The field of metrology is concerned with measuring the quality of manufactured objects. A basic task in metrology is that of determining whether a given manufactured object is of acceptable quality. Usually this involves probing the surface of the object using a measuring device such as a coordinate measuring machine to get a set S of sample points, and then verifying, algorithmically, how well S approximates an ideal object.

A special case of this problem is determining whether an object is *round*, or *circle like*. For our purposes, an object I is *good* if the boundary of I can be contained in an annulus of inner radius $1 - \epsilon$ and outer radius $1 + \epsilon$, for some quality parameter $\epsilon > 0$, and is *bad* otherwise. See Fig. 1 for examples of good and bad objects. We call this problem the *roundness classification problem.*

Little research has been done on probing strategies for the roundness classification problem. A notable exception is the work by Mehlhorn, Shermer, and Yap [4], in which a probing strategy for manufactured disks is coupled with a roundness testing algorithm. Unfortunately, the procedure described in [4] relies on the assumption that the object I is convex. It is usually not the case that the manufacturing process can guarantee this.

In this paper we describe strategies for testing the roundness of manufactured disks and cylinders. We use the *finger probing* model of Cole and Yap [1]. In this model, the measurement device can identify a point in the interior of I and can probe along any ray originating outside of I, i.e., determine the first point on the

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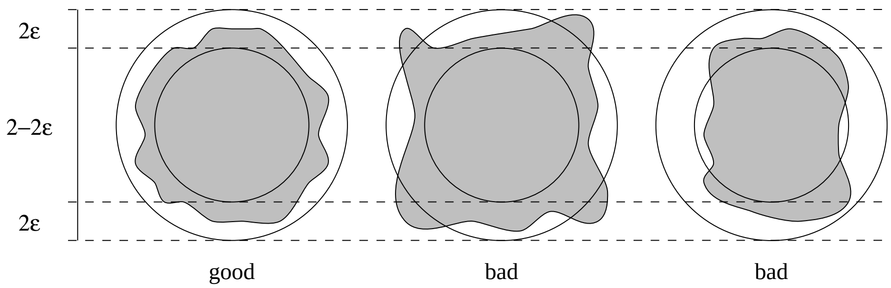

Fig. 1. Examples of good and bad objects.

ray which intersects the boundary of I. The finger probing model is a reasonable abstract model of a coordinate measuring machine [5].

This work extends the results of [4] in several ways. The assumption that the object I is convex is replaced by a much weaker assumption related to visibility. Using this assumption, we give a procedure for testing the roundness of a manufactured disk I using $O(|1/\text{qual}(I)|)$ probes and $O(|1/\text{qual}(I)| \log |1/\text{qual}(I)|)$ computation time. Here $|\text{qual}(I)|$ measures how far the object I is from the boundary between good and bad. For testing the roundness of a manufactured cylinder J we describe a procedure that uses $O(l/\text{qual}(J)^2)$ probes and $O(l/\text{qual}(J)^2 \log(l/\text{qual}(J)^2))$ computation time, where l is the length of J. We also give lower bounds which show that our procedures are optimal, up to constant factors, in terms of the number of probes used.

The remainder of the paper is organized as follows: Section 2 introduces definitions and notation used throughout the remainder of the paper. Section 3 describes procedures for testing the quality of manufactured disks. Section 4 presents a procedure for testing the quality of manufactured cylinders. Section 5 gives lower bounds on the number of probes needed to solve these problems.

2 Definitions, Notation, and Assumptions

In this section, we introduce definitions and notation used throughout the remainder of this paper, and state the assumptions we make on the object being tested. For the most part, notation and definitions are consistent with [4].

For a point p, we use the notation $\text{x}(p)$, $\text{y}(p)$, and $\text{z}(p)$ to denote the x, y, and z coordinates of p, respectively. The letter O is used to denote the origin of the coordinate system. We use the notation $\text{dist}(a, b)$ to denote Euclidean distance between two objects. When a and b are sets of points, $\text{dist}(a, b)$ is the minimum distance between all pairs of points in a and b. The angle formed by three points a, b, and c, is denoted by $\angle abc$, and we always mean the smaller angle unless stated otherwise.

A *planar object* I is defined to be any compact simply connected subset of the plane, with boundary denoted by $\text{bd}(I)$. For a point p, we use $R(p, I)$ and

$r(p, I)$ to denote the maximal and minimal distance, respectively, from p to a point in $\mathrm{bd}(I)$. I.e.,

$$R(p, I) = \max\{\mathrm{dist}(p, p') : p' \in \mathrm{bd}(I)\} \tag{1}$$
$$r(p, I) = \min\{\mathrm{dist}(p, p') : p' \in \mathrm{bd}(I)\}\ . \tag{2}$$

For a point p, let

$$\mathrm{qual}(p, I) = \min\{r(p, I) - (1 - \epsilon), (1 + \epsilon) - R(p, I)\} \tag{3}$$
$$\mathrm{qual}(I) = \max\{\mathrm{qual}(p, I) : p \in \mathbb{R}^2\}\ . \tag{4}$$

Any point c_I with $\mathrm{qual}(c_I, I) = \mathrm{qual}(I)$ is called a *center* of I. The value $\mathrm{qual}(I)$ is called the *quality* of the object I. An object I with $\mathrm{qual}(I) > 0$ is *good* while an object I with $\mathrm{qual}(I) < 0$ is *bad*. A procedure which determines whether a planar object is good or bad is called a *roundness classification procedure*.

In order to have a testing procedure which is always correct and which terminates, it is necessary to make some assumptions about the object I being tested. The following assumption made in [4] is referred to as the *minimum quality assumption*, and refers to the fact that the manufacturing process can guarantee that manufactured objects have a minimum quality (although perhaps not enough to satisfy our roundness criterion).

Assumption 1. $R(c_I, I) \leq 1 + \delta$ and $r(c_I, I) \geq 1 - \delta$, for some constant $0 < \delta < 1/21$, i.e., the boundary of I is contained in an annulus of inner radius $1 - \delta$ and outer radius $1 + \delta$.

The minimum quality assumption alone is not sufficient. If the object under consideration contains oddly shaped recesses, then it may be the case that these recesses cannot be found using finger probes. We say that an object I is *star-shaped* if there exists a point $k \in I$ such that for any point $p \in I$, the line segment joining k and p is a subset of I. We call the set of all points with this property the *kernel* of I. The following assumption ensures that all points in $\mathrm{bd}(I)$ can be probed by directing probes close to the center of I.

Assumption 2. I is a star-shaped object, and its kernel contains all points p such that $\mathrm{dist}(c_I, p) \leq \alpha$, for some constant $1 - \delta > \alpha > 2\delta$.

We observe that our assumptions are weaker than those in [4].

Observation 1. *The set of convex objects satisfying Assumption 1 is strictly contained in the set of objects satisfying Assumptions 1 and 2.*

3 Testing Disks

3.1 The Simplified Procedure

In this section we describe a simplified testing procedure which assumes that we know the object being tested is centered at the origin, O. The motivation for describing this simplified procedure is pedagogical; it is a simple example which helps in understanding the full procedure.

Our testing procedure tests the roundness of an object I by taking a set S of probes at uniform intervals directed at the origin. We use the notation probe(n, p) to denote the set of points obtained by taking n probes directed at the point p in directions $2\pi/n, 4\pi/n, \ldots, 2(n-1)\pi/n$. The procedure repeatedly doubles the size of the sample until either (1) a set of sample points is found which cannot be covered by an annulus of inner radius $1-\epsilon$ and outer radius $1+\epsilon$, in which case I is rejected, or (2) the set of sample points can be covered by an annulus with inner radius sufficiently larger than $1-\epsilon$ and outer radius sufficiently smaller than $1+\epsilon$, in which case we can be sure that I is a good object.

Procedure 1 Tests the roundness of the object I centered at the origin.

```
r ← 1
R ← 1
n ← n_0
Δ ← f(n)
repeat
  S ← probe(n, O)
  if ∃p ∈ S : dist(p, O) > 1 + ε or dist(p, O) < 1 − ε then
    return REJECT
  end if
  r ← 1 − ε + Δ
  R ← 1 + ε − Δ
  n ← 2n
  Δ ← f(n)
until ∀p ∈ S : dist(p, O) < R and dist(p, O) > r
return ACCEPT
```

The function $f(n)$ which appears in the procedure is defined as

$$f(n) = \frac{1}{n}\left(\frac{(1+\delta)^2(\alpha^2+1+2\delta+\delta^2)}{\alpha^2}\right)^{\frac{1}{2}} \le \frac{12}{n} \ . \tag{5}$$

and the constant n_0 is defined as

$$n_0 = \lceil \pi / \arctan(\alpha/(1+\delta)) \rceil \le 70 \ . \tag{6}$$

With these definitions, we obtain the following crucial lemma.

Lemma 1. *Let I be a planar object with center c_I. Let S be the set of results of $n \ge n_0$ probes directed at c_I in directions $0, 2\pi/n, 4\pi/n, \ldots, 2\pi(n-1)/n$. Then for any point $p \in \mathrm{bd}(I)$, there exists a point $p' \in S$ such that $\mathrm{dist}(p, p') \le f(n)$.*

Proof. Assume wlog that $c_I = O$, $\mathrm{x}(p) = 0$, and $1 - \delta \le \mathrm{y}(p) \le 1 + \delta$. We will upper-bound $|\mathrm{x}(p) - \mathrm{x}(p')|$ and $|\mathrm{y}(p) - \mathrm{y}(p')|$. Refer to Fig. 2 for an illustration.

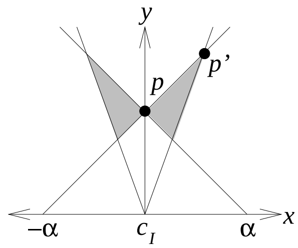

Fig. 2. Constraints on the position of p'. The point p' must be in the shaded region, and $\mathrm{dist}(p, p')$ is maximized when p' is placed as shown.

First note that there exists a sample point $p' \in S$ such that $0 \leq \angle p c_I p' \leq \pi/n$. By Assumption 1, $\mathrm{dist}(O, p') \leq 1 + \delta$, so an upper bound on $|\mathrm{x}(p) - \mathrm{x}(p')|$ is

$$|\mathrm{x}(p) - \mathrm{x}(p')| = |\mathrm{x}(p')| \leq (1+\delta)\sin(\pi/n) \quad (7)$$
$$\leq (1+\delta)\pi/n \quad (8)$$

Since $\angle p c_I p' \leq \pi/n$, p' must lie in the cone defined by the inequality

$$\mathrm{y}(p') \geq |\mathrm{x}(p')| \left(\frac{\cos(\pi/n)}{\sin(\pi/n)} \right) . \quad (9)$$

Next we note that the slope of the line through p' and p must be in the range $[-\mathrm{y}(p)/\alpha, \mathrm{y}(p)/\alpha]$, otherwise Assumption 2 is violated. If $n \geq n_0$, then the region in which p' can be placed is bounded, and $|\mathrm{y}(p) - \mathrm{y}(p')|$ is maximized when p' lies on one of the bounding lines

$$f_l(x) = x\mathrm{y}(p)/\alpha + \mathrm{y}(p) \quad (10)$$
$$f_r(x) = -x\mathrm{y}(p)/\alpha + \mathrm{y}(p) \quad (11)$$

Since both lines are symmetric about $x = 0$, assume that $\mathrm{x}(p')$ lies on f_l, giving

$$|\mathrm{y}(p) - \mathrm{y}(p')| \leq |\mathrm{y}(p) - f_l(\mathrm{x}(p'))| \quad (12)$$
$$= |\mathrm{x}(p')\mathrm{y}(p)/\alpha| \quad (13)$$
$$\leq |\mathrm{x}(p')(1+\delta)/\alpha| \quad (14)$$
$$\leq (1+\delta)^2\pi/\alpha n \quad (15)$$

Substituting (8) and (15) into the Euclidean distance formula and simplifying yields the stated inequality. □

Theorem 1. *There exists a roundness classification procedure that can correctly classify any planar object I with center $c_I = O$ and satisfying Assumptions 1 and 2 using $O(|1/\mathrm{qual}(I)|)$ probes and $O(|1/\mathrm{qual}(I)|)$ computation time.*

Proof. We begin by showing that the procedure is correct. We need to show that the procedure never rejects a good object and never accepts a bad object. The former follows from the fact that the procedure only ever rejects an object when it finds a point on the object's boundary which is not contained in the annulus of inner radius $1-\epsilon$ and outer radius $1+\epsilon$ centered at c_I.

Next we prove that the procedure never accepts an bad object. Lemma 1 shows that there is no point in $\mathrm{bd}(I)$ which is of distance greater than $f(n)$ from all points in S. The procedure only accepts I when all points in S are of distance at least $\Delta = f(n)$ from the boundary of the annulus of inner radius $1-\epsilon$ and outer radius $1+\epsilon$ centered at O. Therefore, if the procedure accepts I, all points in $\mathrm{bd}(I)$ are contained in an annulus of inner radius $1-\epsilon$ and outer radius $1+\epsilon$, i.e., the object is good.

Next we prove that the running time is $O(|1/\mathrm{qual}(I)|)$. First we observe that $f(n) \in O(1/n)$. Next, note that the computation time and number of probes used during each iteration is linear with respect to the value of n, and the value of n doubles after each iteration. Thus, asymptotically, the computation time and number of probes used are dominated by the value of n during the last iteration. There are two cases to consider.

Case 1: Procedure 1 accepts I. In this case, the procedure will certainly terminate once $\Delta \le \mathrm{qual}(I)$. This takes $O(\log(1/\mathrm{qual}(I)))$ iterations. During the final iteration, $n \in O(1/\mathrm{qual}(I))$.

Case 2: Procedure 1 rejects I. In this case, there is a point on $\mathrm{bd}(I)$ at distance $\mathrm{qual}(I)$ outside the circle with radius $1+\epsilon$ centered at O, or there is a point in $\mathrm{bd}(I)$ at distance $\mathrm{qual}(I)$ inside of the circle with radius $1-\epsilon$ centered at O. In either case, Lemma 1 ensures that the procedure will find a bad point within $O(\log|1/\mathrm{qual}(I)|)$ iterations. During the final iteration, $n \in O(|1/\mathrm{qual}(I)|)$. □

3.2 The Full Procedure

The difficulty in implementing Procedure 1 is that we may not know the position of the exact center, c_I, of I. However, the following result from [4] allows us to use this procedure anyhow.

Theorem 2 (Near-Center). *Let I be a planar object with center c_I and which satisfies Assumption 1. Then 6 probes and constant computation time suffice to determine a point c_0 such that* $\mathrm{dist}(c_I, c_0) \le 2\delta$.

We call any such point c_0 a *near-center* of I. As the following lemmata show, knowing a near center is almost as useful as knowing the true center. Before we state the lemma, we need the following definitions.

$$f'(n) = \frac{1}{n}\left((1+3\delta)^2\pi^2 + \frac{(1+3\delta)^4\pi^2}{(\alpha-2\delta)^2}\right)^{\frac{1}{2}} \tag{16}$$

$$n'_0 = \lceil \pi / \arctan(\alpha/(1+3\delta)) \rceil \tag{17}$$

Lemma 2. *Let I be a planar object with center c_I and near-center c_0. Let S be the set of results of $n \geq n_0'$ probes directed at c_0 in directions $0, 2\pi/n, 4\pi/n, \ldots, 2\pi(n-1)/n$. Then for any point $p \in \mathrm{bd}(I)$, there exists a point $p' \in S$ such that $\mathrm{dist}(p, p') \leq f'(n)$.*

Proof. The proof is almost a verbatim translation of the proof of Lemma 1, except that we assume that $c_0 = O$. With this assumption we derive the bounds

$$|\mathrm{x}(p) - \mathrm{x}(p')| \leq (1 + 3\delta)(\pi/n) \tag{18}$$

$$|\mathrm{y}(p) - \mathrm{y}(p')| \leq (1 + 3\delta)^2\pi/n(\alpha - 2\delta)| \tag{19}$$

Substituting these values into the formula for the Euclidean distance and simplifying yields the desired result. □

Lemma 3. *Let I be a planar object with center c_I and near-center c_0. Let S be the set of results of n probes directed at c_0 in directions $0, 2\pi/n, 4\pi/n, \ldots, 2\pi(n-1)/n$, and let c_S be the center of S. Then*

$$R(c_S, S) \leq R(c_S, I) \leq R(c_S, S) + f'(n) \tag{20}$$

$$r(c_S, S) - f'(n) \leq r(c_S, I) \leq r(c_S, S)\ . \tag{21}$$

Proof. We prove only the bounds on the $R(c_S, I)$ as the proof of the bounds on $r(c_S, I)$ are symmetric. The lower bound on $R(c_S, I)$ is immediate, since $S \subset \mathrm{bd}(I)$. To see the upper bound, choose any point $p \in \mathrm{bd}(I)$ such that $\mathrm{dist}(c_S, p) = R(c_S, I)$. By Lemma 1 there exists $p' \in S$ such that $\mathrm{dist}(p', p) \leq f'(n)$. Therefore $\mathrm{dist}(c_S, p) \leq \mathrm{dist}(c_S, p') + f'(n)$, which implies that $R(c_S, I) \leq R(c_S, S) + f'(n)$. □

Lemma 4. *Let I be a planar object with center c_I and near-center c_0. Let S be the set of results of n probes directed at c_0 in directions $0, 2\pi/n, 4\pi/n, \ldots, 2\pi(n-1)/n$. Then $\mathrm{qual}(S) - f'(n) \leq \mathrm{qual}(I) \leq \mathrm{qual}(S)$*

Proof. The proof can be found in [4].

Theorem 3. *There exists a roundness classification procedure that can correctly classify any planar object I satisfying Assumptions 1 and 2 using $O(1/|\mathrm{qual}(I)|)$ probes and $O(|1/\mathrm{qual}(I)| \log |1/\mathrm{qual}(I)|)$ computation time.*

Proof. We make the following modifications to Procedure 1. In Line 3, we set the value of n to n_0'. In Lines 4 and 13, we replace $f(n)$ with $f'(n)$. In Line 6 we directed our probes at c_0 rather than O. In Lines 7 and 14, we replace the simple test with a call to one of the $O(n \log n)$ time referenced roundness algorithms in [2] or [3], to test whether the sample set S can be covered by an annulus with the specified inner and outer radius.

Lemma 4 ensures that the procedure never accepts a bad object and never rejects a good object. i.e., the procedure is correct. The procedure terminates once $f'(n) < |\mathrm{qual}(I)|$. This happens after $O(\log |1/qual(I)|)$ iterations, at which point $n \in O(|1/\mathrm{qual}(I)|$. □

4 Testing Cylinders

Before we can describe a quality testing procedure for cylinders, we must generalize the notion of quality to cylinders. In this work, we are concerned with the roundess of cylinders, and not their height, or the flatness of their ends. For these reasons, we will assume that our manufactured cylinders have length l, and that their ends are perfectly flat. The object, J, that we are interested in testing is a compact simply connected subset of the space $(x, y, [0, l])$.

We assume that J is resting on the (x, y) plane and that we know the orientation of the cylinder. Define J_h to be the set of all points (x, y) such that $(x, y, h) \in J$. Note that J_h is a planar object. We define the *outer boundary* of J as $J_{\text{out}} = \bigcup_{0 \le h \le l} J_h$, and we define the *inner boundary* of J as $J_{\text{in}} = \bigcap_{0 \le h \le l} J_h$. For a point p on the plane, we use $R(p, J)$ and $r(p, J)$ to denote the maximal and minimal distance, respectively, from p to a point in $\text{bd}(J_{\text{out}})$ and $\text{bd}(J_{\text{in}})$, respectively. For a point p, let

$$\text{qual}(p, J) = \min\{r(p, J) - (1 - \epsilon), (1 + \epsilon) - R(p, J)\} \tag{22}$$

$$\text{qual}(J) = \max\{\text{qual}(p, J) : p \in \mathbb{R}^3\} \ . \tag{23}$$

We call any point c_J with $\text{qual}(c_J, J) = \text{qual}(J)$ a *center* of J. Note that according to these definitions J is an good object if there exists an annulus of inner radius $1 - \epsilon$ and outer radius $1 + \epsilon$ which covers both $\text{bd}(J_{\text{in}})$ and $\text{bd}(J_{\text{out}})$.

We require the following minimum quality assumptions.

Assumption 3. $R(c_J, J) \le 1 + \delta$ and $r(c_J, J) \ge 1 - \delta$, for some constant $0 < \delta < 1/21$.

Assumption 4. For all $0 \le h \le l$, J_h is a star-shaped object, and its kernel contains all points p such that $\text{dist}(c_J, p) \le \alpha$, for some constant $1 - \delta > \alpha > 2\delta$.

Assumption 5. Let J be an object with center c_J, and let $J[h-\alpha, h+\alpha]$ be the subset of points in J with z coordinate in $[h-\alpha, h+\alpha]$. Let K be the intersection of the infinite length cylinder of radius α, which is perpendicular to the (x, y) plane and is centered at O with $J[h - \alpha, h + \alpha]$. Then $J[h - \alpha, h + \alpha]$ is a star shaped object with kernel K.

Note that Assumptions 3 and 4 allow us to find a near-center c_0 of J in constant time. Our procedure for testing cylinders is exactly the same as the procedure for testing disks described in Sect. 3.2, except that during each iteration, we perform $ln/2\pi$ sets of probes along the planes $z = 0, z = 2\pi/n, z = 4\pi/n, \ldots, z = l$, where each set contains n probes directed at c_0. Note that the number of probes performed is $O(ln^2)$. After collecting these sample points, they are projected onto the (x, y)-plane, and the algorithm of [2] or [3], determine whether there exists an annulus of inner radius r and outer radius R which contains them.

As in Sect. 3, let us define the function

$$f''(n) = f'(n) + \frac{1}{n}\left(((1 + 3\delta)\pi/\alpha)^2 + \pi^2\right)^{\frac{1}{2}} \ . \tag{24}$$

Lemma 5. *Let J be an object of length l with center c_J, and satisfying Assumptions 3, 4, and 5. Let c_0 be any point such that* $\mathrm{dist}(c_I, c_0) \leq 2\delta$, *and let S be a set of $ln/2\pi$, $n \geq n_0'$, probes directed at c_0 as described above. Then for any point $p \in \mathrm{bd}(J)$ there exists a point $p' \in S$ such that* $\mathrm{dist}(p, p') \leq f''(n)$.

Proof (Sketch). Note that S contains a ring of probes S' such that for all $p' \in S$, $|z(p') - z(p)| \leq \pi/n$. With reference to Fig. 3, one can prove that $\mathrm{dist}(p, p'') \leq f'(n)$ using the same arguments used in the proof of Lemma 2. Using Assumption 5, once can also prove that $\mathrm{dist}(p'', p') \leq \frac{1}{n}\left(((1+3\delta)\pi/\alpha)^2 + \pi^2\right)^{\frac{1}{2}}$. The stated bound then follows from the triangle inequality. □

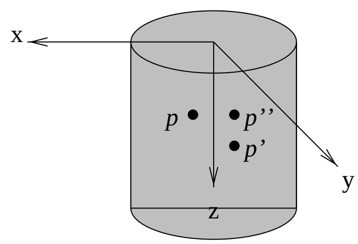

Fig. 3. The proof of Lemma 5.

With this result, and following the arguments of Sect. 3.2, it is not difficult to prove the correctness and running time of the testing procedure for cylinders.

Theorem 4. *There exists a roundness classification procedure for cylinders that can correctly classify any object J of length l and satisfying Assumptions 3, 4, and 5 using $O(l/\mathrm{qual}(J)^2)$ probes and $O(l/\mathrm{qual}(J)^2 \log(l/\mathrm{qual}(J)^2))$ computation time.*

5 Lower Bounds

In this section, we give lower bounds which show that the number of probes used in our testing procedures is optimal up to a constant factor. These lower bounds hold even if the center of the object is known in advance. We begin with a lower bound for planar objects.

Theorem 5. *Any roundness classification procedure that is always correct requires, in the worst case, $\Omega(|1/\mathrm{qual}(I)|)$ probes to classify a planar object I with center $c_I = O$ and satisfying Assumption 1 and Assumption 2.*

Proof (Sketch). Let I and I' be the two objects depicted in Fig. 4. For any $0 \leq \psi \leq \epsilon$, $\mathrm{qual}(I) = -\mathrm{qual}(I') = \psi$. It is not difficult to see that if a procedure uses $o(|1/\psi|)$ probes, then by rotating I' appropriately, it is possible to "hide" the recess in I' so that the procedure cannot differentiate between I and I'. □

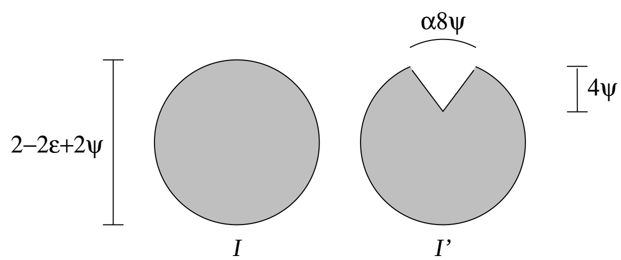

Fig. 4. The proof of Theorem 5.

A similar argument can be used for cylinders. The difference being that the recess in the object J' is a circular cone. That the recess in J' can be hidden comes from the fact that a set of n^2 points in the unit square always contains an empty circle of radius $\Omega(1/n)$.[1]

Theorem 6. *Any roundness classification procedure for cylinders that is always correct requires, in the worst case,* $\Omega(l/\mathrm{qual}(J)^2)$ *probes to classify an object* J *with center* $c_I = O$ *and satisfying Assumptions 3, 4, and 5.*

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[1] To see this, divide the square into a $(n+1) \times (n+1)$ grid. Some cells in this grid must have no points in them, therefore an empty circle of diameter $1/(n+1)$ can be placed in one of these empty cells.