COMP 2804 — Solutions Assignment 1

Question 1:

- Write your name and student number.

Solution:

- Name: Kylian Mbappé

- Student number: 7

Question 2: If n and d are positive integers, then d is a *divisor* of n, if n/d is an integer. Determine the number of divisors of the integer

$$1,170,725,783,076,864 = 2^{17} \cdot 3^{12} \cdot 7^5.$$

Solution: The main observation is that every divisor of

$$2^{17} \cdot 3^{12} \cdot 7^5$$

is of the form

$$2^a \cdot 3^b \cdot 7^c,$$

for some $a \in \{0, 1, 2, \ldots, 17\}$, $b \in \{0, 1, 2, \ldots, 12\}$, and $c \in \{0, 1, 2, \ldots, 5\}$. (Note that 0 is included!) Therefore, in order to specify one divisor, we must specify a, b, and c.

Procedure: Specify one divisor.

Task 1: Choose $a \in \{0, 1, 2, \ldots, 17\}$. There are 18 ways to do this.

Task 2: Choose $b \in \{0, 1, 2, \ldots, 12\}$. There are 13 ways to do this.

Task 3: Choose $c \in \{0, 1, 2, \ldots, 5\}$. There are 6 ways to do this.

By the Product Rule, the total number of ways to do the procedure is equal to

$$18 \cdot 13 \cdot 6 = 1404.$$

This is also the number of divisors.

Question 3: Consider permutations of the set $\{a, b, c, d, e, f, g\}$ that do not contain bge (in this order) and do not contain eaf (in this order). Prove that the number of such permutations is equal to 4806.

(You must use counting rules that we have seen in class.)

Solution:

1. Let U be the set of all permutations of the set $\{a, b, c, d, e, f, g\}$.

2. Let A be the set of all permutations in U that contain bge (in this order).

3. Let B be the set of all permutations in U that contain eaf (in this order).
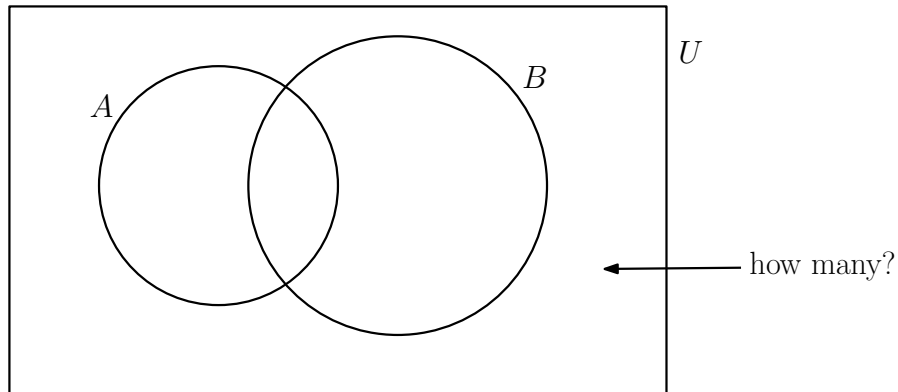
By looking at the Venn diagram, we see that the question asks for the number of permutations in U that are not in A and are not in B. In other words, we are asked to determine

$$|U| - |A \cup B|.$$

1. Since the set $\{a, b, c, d, e, f, g\}$ has 7 elements,

$$|U| = 7! = 5040.$$

2. What is the size of A: Imagine bge to be one symbol, say X. Then A is the set of all permutations of $\{a, c, d, f, X\}$. Therefore,

$$|A| = 5! = 120.$$

Here is a different way to determine the size of A: The permutation will contain bge. We have to place the letters a, c, d, f. Some of these four letters will go to the left of bge, and some of them will go to the right. Imagine the space to the left of bge to be a bookshelf. Also, imagine the space to the right bge to be a bookshelf. Then we have 2 bookshelves and have to place the four "books" a, c, d, f.

In class, we have seen the following: If we have M books and N bookshelves, then there are
$$\frac{(N + M - 1)!}{(N - 1)!}$$
ways to place the books. In our case, $M = 4$ and $N = 2$.

3. By the same reasoning,
$$|B| = 5! = 120.$$

4. What is the size of $A \cap B$? Note that this is the set of all permutations of $\{a, b, c, d, e, f, g\}$ that contain $bgeaf$ (in this order). Imagine $bgeaf$ to be one symbol, say Y. Then $A \cap B$ is the set of all permutations of $\{c, d, Y\}$. Therefore,

$$|A \cap B| = 3! = 6.$$

Of course, we get the same answer if we use the books-on-shelves approach.

5. Using Inclusion-Exclusion, we get

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
&= 120 + 120 - 6 \\
&= 234.
\end{aligned}
$$

6. Thus, the final answer to the question is

$$|U| - |A \cup B| = 5040 - 234 = 4806.$$

Question 4: Let $n \geq 12$ be an integer and let $\{B_1, B_2, \ldots, B_n\}$ be a set of n beer bottles. Consider permutations of these bottles such that there are exactly 10 bottles between B_1 and B_n. (B_1 can be to the left or right of B_n.) Prove that the number of such permutations is equal to

$$\binom{n-2}{10} \cdot 10! \cdot 2 \cdot (n-11)!.$$

Solution: We are going to use the Product Rule:

Task 1: Choose a subset of 10 bottles from the set $\{B_2, B_3, \ldots, B_{n-1}\}$. There are $\binom{n-2}{10}$ ways to do this.

Task 2: Place the 10 chosen bottles from Task 1 in some order. There are $10!$ ways to do this.

Task 3: Choose one of the following two options:
Option 1: Place B_1 to the left of the 10 bottles and place B_n to the right of the 10 bottles.
Option 2: Place B_n to the left of the 10 bottles and place B_1 to the right of the 10 bottles.
There are 2 ways to do this.

Task 4: At this moment, we have placed 12 bottles. Imagine these 12 bottles to be one big bottle. In this fourth task, we take a permutation of the big bottle and the remaining $n - 12$ bottles. There are $(n-11)!$ ways to do this.

By the Product Rule, the total number of ways to do the entire procedure is equal to

$$\binom{n-2}{10} \cdot 10! \cdot 2 \cdot (n-11)!.$$

Alternative way to do Task 4: We have already placed 12 bottles, and we have to place the remaining $n-12$ bottles. Some of these $n-12$ bottles will be placed to the left of the 12 bottles, and some of them will be placed to the right. Imagine the space to the left of the 12 bottles to be one beershelf. Also, imagine the space to the right of the 12 bottles to be one beershelf. Then we have 2 beershelves and have to place $n-12$ beer bottles on them.

In class, we have seen the following[1]: If we have M beer bottles and N beershelves, then there are
$$\frac{(N+M-1)!}{(N-1)!}$$
ways to place the bottles. In our case, $M = n - 12$ and $N = 2$.

Question 5: Let $n \geq 3$ be an integer. The Gn (or Group of n) is an international forum where the n leaders of the world meet to drink beer together. Two of these leaders are Donald Trump and Justin Trudeau. At the end of their meeting, the n leaders stand on a line and a group photo is taken.

- Determine the number of ways in which the n leaders can be arranged on a line, if Donald Trump and Justin Trudeau are standing next to each other.

  Solution: Imagine DT and JT to be one person, say X. Then, the number of people is equal to $n-1$.

  Task 1: Take a permutation of these $n-1$ people. There are $(n-1)!$ ways to do this.

  Task 2: Replace X by DT,JT or by JT,DT. There are 2 ways to do this.

  By the Product Rule, the number of arrangements is equal to

  $$2 \cdot (n-1)!.$$

- Determine the number of ways in which the n leaders can be arranged on a line, if Donald Trump and Justin Trudeau are not standing next to each other.

  Solution: The total number of possible permutations is equal to $n!$. If we subtract from this the answer to the previous part, then we have counted all permutations in which DT and JT are not standing next to each other. Thus, the answer to this part is
  $$n! - 2 \cdot (n-1)! = (n-2) \cdot (n-1)!.$$

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[1] In class, we did it for books, but it works for beer bottles too!

- Determine the number of ways in which the n leaders can be arranged on a line, if Donald Trump is to the left of Justin Trudeau. (Donald does not necessarily stand immediately to the left of Justin.)

First Solution:

Task 1: Choose 2 positions, out of n positions. There are $\binom{n}{2}$ ways to do this.

Task 1: Place DT at the leftmost of the chosen positions. There is 1 way to do this.

Task 2: Place JT at the rightmost of the chosen positions. There is 1 way to do this.

Task 3: Place the remaining $n-2$ people in the remaining $n-2$ positions. There are $(n-2)!$ ways to do this.

By the Product Rule, the answer is

$$\binom{n}{2} \cdot (n-2)! = \frac{n!}{2}.$$

Second Solution:

Consider a table T that contains all permutations in which DT is to the left of JT.

Make a copy T' of T, but write all permutations in reverse order. Then T' contains all permutations in which DT is to the right of JT.

Observe that if we combine T and T' into one big table, then we obtain all $n!$ permutations.

Let x be the number of permutations in T. Then the arguments above imply that

$$2x = n!$$

and, therefore,

$$x = \frac{n!}{2}.$$

Question 6: A *flip* in a bitstring is a pair of adjacent bits that are not equal. For example, the bitstring 010011 has three flips: The first two bits form a flip, the second and third bits form a flip, and the fourth and fifth bits form a flip.

- Determine the number of bitstrings of length 7 that have exactly 3 flips at the following positions: The second and third bits form a flip, the third and fourth bits form a flip, and the fifth and sixth bits form a flip.

- Let $n \geq 2$ and k be integers with $0 \leq k \leq n-1$. Determine the number of bitstrings of length n that have exactly k flips.

Solution: We start with the first part. Refer to the figure below. The main observation is the following: If we fix the first bit, then the entire bitstring of length 7 is fixed.
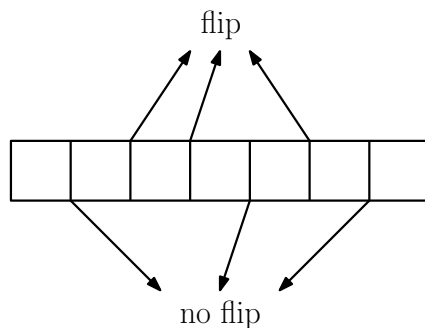
1. Assume the first bit is 0. Then the second bit must be 0, because there is no flip. The third bit must be 1, because there is a flip. If we continue in this way, we get the bitstring

$$0010011.$$

2. Assume the first bit is 1. Then the second bit must be 1, because there is no flip. The third bit must be 0, because there is a flip. If we continue in this way, we get the bitstring

$$1101100.$$

Thus, the answer to this part is 2.

For the second part: A flip is defined by a pair of two consecutive bits. The number of such consecutive pairs is $n - 1$.

Task 1: Out of the $n - 1$ consecutive pairs, choose a subset of size k. There are $\binom{n-1}{k}$ ways to do this.

Task 2: For the k consecutive pairs chosen in Task 1, there are 2 bitstrings of length n that have flips at exactly these k pairs. In this second task, we choose one of them. There are 2 ways to do this.

By the Product Rule, the number of bitstrings of length n that have exactly k flips is equal to

$$2\binom{n - 1}{k}.$$

Question 7: Consider 10 male students $M_1, M_2, \ldots, M_{10}$ and 7 female students $F_1, F_2, \ldots, F_7$. Assume these 17 students are arranged on a horizontal line such that no two female students are standing next to each other. We are interested in the number of such arrangements, where the order of the students matters.

Explain what is wrong with the following argument:

We are going to use the Product Rule:

- Task 1: Arrange the 7 females on a line. There are 7! ways to do this.

- Task 2: Choose 6 males. There are $\binom{10}{6}$ ways to do this.

- Task 3: Place the 6 males chosen in Task 2 in the 6 "gaps" between the females. There are 6! ways to do this.

- Task 4: At this moment, we have arranged 13 students on a line. We are left with 4 males that have to be placed.

  - Task 4.1: Place one male. There are 14 ways to do this.
  - Task 4.2: Place one male. There are 15 ways to do this.
  - Task 4.3: Place one male. There are 16 ways to do this.
  - Task 4.4: Place one male. There are 17 ways to do this.

By the Product Rule, the total number of ways to arrange the 17 students is equal to

$$7! \cdot \binom{10}{6} \cdot 6! \cdot 14 \cdot 15 \cdot 16 \cdot 17 = 43,528,181,760,000.$$

Solution: In order to apply the Product Rule to a counting problem, we need the following:

1. Phrase the counting problem in terms of doing a procedure, consisting of a number of tasks.

2. There must be a one-to-one correspondence between the different ways to do the procedure and the objects we are counting. In other words:

   (a) Each way to do the procedure must correspond to a unique object we are counting.
   (b) Conversely, each object we are counting must correspond to a unique way to do the procedure.

For this question, property 2 does not hold: We are going to show that the following arrangement can be obtained by doing the procedure in two different ways:

$$F_1 M_1 M_2 F_2 M_3 F_3 M_4 F_4 M_5 F_5 M_6 F_6 M_7 F_7 M_8 M_9 M_{10} \tag{1}$$

Thus, counting the number of ways to do the procedure is not the same as counting the number of arrangements.

First way:

Task 1: Arrange the females as

$$F_1 F_2 F_3 F_4 F_5 F_6 F_7.$$

Task 2: Choose $\{M_1, M_3, M_4, M_5, M_6, M_7\}$.

Task 3: Place the chosen males as follows:

$$F_1 M_1 F_2 M_3 F_3 M_4 F_4 M_5 F_5 M_6 F_6 M_7 F_7.$$

Task 4: We still have to place M_2, M_8, M_9, and M_{10}:

1. Place M_2 between M_1 and F_2.

2. Place M_8 at the end.

3. Place M_9 at the end.

4. Place M_{10} at the end.

This gives the arrangement in (1).

Second way:

Task 1: As above: Arrange the females as

$$F_1 F_2 F_3 F_4 F_5 F_6 F_7.$$

Task 2: Choose $\{M_2, M_3, M_4, M_5, M_6, M_7\}$.

Task 3: Place the chosen males as follows:

$$F_1 M_2 F_2 M_3 F_3 M_4 F_4 M_5 F_5 M_6 F_6 M_7 F_7.$$

Task 4: We still have to place M_1, M_8, M_9, and M_{10}:

1. Place M_1 between F_1 and M_2.

2. Place M_8 at the end.

3. Place M_9 at the end.

4. Place M_{10} at the end.

This again gives the arrangement in (1).

By the way, the correct answer is

$$10! \cdot \binom{11}{7} \cdot 7! = 6,035,420,160,000.$$

You can find the proof in the solutions for Assignment 1, Fall 2018.

Question 8: Let $n \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$. An ordered triple (A, x, y) is called *awesome*, if (i) $A \subseteq S$, (ii) $x \in A$, and (iii) $y \in A$.

- Let k be an integer with $1 \leq k \leq n$. Determine the number of awesome triples (A, x, y) with $|A| = k$.

  Solution: We are going to use the Product Rule:

  Task 1: Choose a subset A of size k. There are $\binom{n}{k}$ ways to do this.

  Task 2: Choose an element x in A. There are k ways to do this.

  Task 3: Choose an element y in A. There are k ways to do this.

  By the Product Rule, the number of ways to do the entire procedure is equal to

  $$k^2 \binom{n}{k}.$$

  This is equal to the number of awesome triples (A, x, y) with $|A| = k$.

- Prove that the number of awesome triples (A, x, y) with $x = y$ is equal to

  $$n \cdot 2^{n-1}.$$

  Solution: We are going to use the Product Rule:

  Task 1: Choose an element x in S. There are n ways to do this.

  Task 2: Take y to be equal to x. There is 1 way to do this.

  Task 3: Choose an arbitrary subset A' of $S \setminus \{x\}$. There are 2^{n-1} ways to do this.

  Task 4: Let $A = A' \cup \{x\}$. Then (A, x, x) is an awesome triple. There is 1 way to do this.

  By the Product Rule, the number of ways to do the entire procedure is equal to

  $$n \cdot 2^{n-1}.$$

  This is equal to the number of awesome triples (A, x, y) with $x = y$.

- Determine the number of awesome triples (A, x, y) with $x \neq y$.

  Solution: We are going to use the Product Rule:

  Task 1: Choose an element x in S. There are n ways to do this.

  Task 2: Choose an element y in $S \setminus \{x\}$. There are $n - 1$ ways to do this.

  Task 3: Choose an arbitrary subset A' of $S \setminus \{x, y\}$. There are 2^{n-2} ways to do this.

  Task 4: Let $A = A' \cup \{x, y\}$. Then (A, x, y) is an awesome triple. There is 1 way to do this.

  By the Product Rule, the number of ways to do the entire procedure is equal to

  $$n(n - 1) \cdot 2^{n-2}.$$

  This is equal to the number of awesome triples (A, x, y) with $x \neq y$.

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- Use the above results to prove that

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1}.$$

Solution: We are going to count the awesome triples in two different ways.

First, every awesome triple (A, x, y) either has $x = y$ or $x \neq y$. Thus, the total number of awesome triples is equal to the sum of the answers to parts 2 and 3:

$$n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1}. \tag{2}$$

For the second way, we divide all awesome triples (A, x, y) into groups, based on the size of A. Note that, since A must contain x, the size of A can be any number in $\{1, 2, \ldots, n\}$.

For each $k = 1, 2, \ldots, n$, group G_k is the set of all awesome triples (A, x, y) for which $|A| = k$. We have seen in the first part that

$$|G_k| = k^2 \binom{n}{k}.$$

By the Sum Rule, the total number of awesome triples is equal to

$$\sum_{k=1}^{n} |G_k| = \sum_{k=1}^{n} k^2 \binom{n}{k}. \tag{3}$$

Since (2) and (3) count the same things, they must be equal.

Question 9: Let S be a set consisting of 19 two-digit integers. Thus, each element of S belongs to the set $\{10, 11, \ldots, 99\}$.

Use the Pigeonhole Principle to prove that this set S contains two distinct elements x and y, such that the sum of the two digits of x is equal to the sum of the two digits of y.

Solution: If $x \in \{10, 11, \ldots, 99\}$, then

$$SumOfDigits(x) \in \{1, 2, \ldots, 18\}.$$

- We will use 18 boxes that are labeled $1, 2, \ldots, 18$.

- Each of the 19 elements x of S is thrown in the box labeled $SumOfDigits(x)$.

- In this way, we throw 19 numbers in 18 boxes. By the Pigeonhole Principle, there is a box that has 2 of these numbers, say x and y. Since x and y are in the same box, we have $SumOfDigits(x) = SumOfDigits(y)$.

Question 10: Let S be a set consisting of 9 people. Every person x in S has an age $age(x)$, which is an integer with $1 \leq age(x) \leq 60$.

- Assume that there are two people in S having the same age. Prove that there exist two subsets A and B of S such that (i) both A and B are non-empty, (ii) $A \cap B = \emptyset$, and (iii) $\sum_{x \in A} age(x) = \sum_{x \in B} age(x)$.

  Solution: Let x and y be two people that have the same age. Then we take $A = \{x\}$ and $B = \{y\}$.

- Assume that all people in S having different ages. Use the Pigeonhole Principle to prove that there exist two subsets A and B of S such that (i) both A and B are non-empty, and (ii) $\sum_{x \in A} age(x) = \sum_{x \in B} age(x)$.

  Solution: If A is a non-empty subset of S, then

  $$\sum_{x \in A} age(x) \geq 1,$$

  because A contains at least one person whose age is at least 1. We also have

  $$\sum_{x \in A} age(x) \leq 60 + 59 + 58 + 57 + 56 + 55 + 54 + 53 + 52 = 504,$$

  because A contains at most 9 elements, and all ages are distinct.

  This implies that for any non-empty subset A,

  $$\sum_{x \in A} age(x) \in \{1, 2, \ldots, 504\}.$$

  We will use 504 boxes that are labeled $1, 2, \ldots, 504$. Each of the $2^9 - 1 = 511$ non-empty subsets A of S is thrown in the box labeled $\sum_{x \in A} age(x)$. In this way, we throw 511 non-empty subsets in 504 boxes. By the Pigeonhole Principle, there is a box that has 2 non-empty subsets, say A and B. Since A and B are in the same box, we have $\sum_{x \in A} age(x) = \sum_{x \in B} age(x)$.

- Assume that all people in S having different ages. Prove that there exist two subsets A and B of S such that (i) both A and B are non-empty, (ii) $A \cap B = \emptyset$, and (iii) $\sum_{x \in A} age(x) = \sum_{x \in B} age(x)$.

  Solution: In the previous part, we have shown that there are two non-empty subsets A and B of S, such that $\sum_{x \in A} age(x) = \sum_{x \in B} age(x)$. Since A and B may not be disjoint, we remove $A \cap B$ from both A and B. Thus, let

  $$A' = A \setminus (A \cap B)$$

  and

  $$B' = A \setminus (A \cap B).$$

Observe that A cannot be a proper subset of B, and B cannot be a proper subset of A. Therefore, both A' and B' are non-empty, $A' \cap B' = \emptyset$, and $\sum_{x \in A'} age(x) = \sum_{x \in B'} age(x)$.