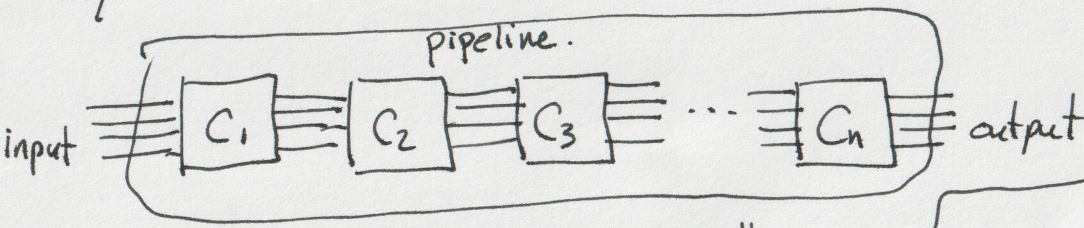
Last class: Independent events, pairwise/mutual independence.

Sequential circuit:



· E_i = "the circuit C_i works properly"

$Pr(A)$ = "the pipeline works properly".

$= Pr(E_1 \cap E_2 \cap E_2 \cap \cdots \cap E_n)$

$= Pr(E_1) \cdot Pr(E_2) \cdot Pr(E_3) \cdots Pr(E_n)$.

mutual indep.

$= \underbrace{p \cdot p \cdot p \cdots p}_{n}$.

$= p^n$.

Given:

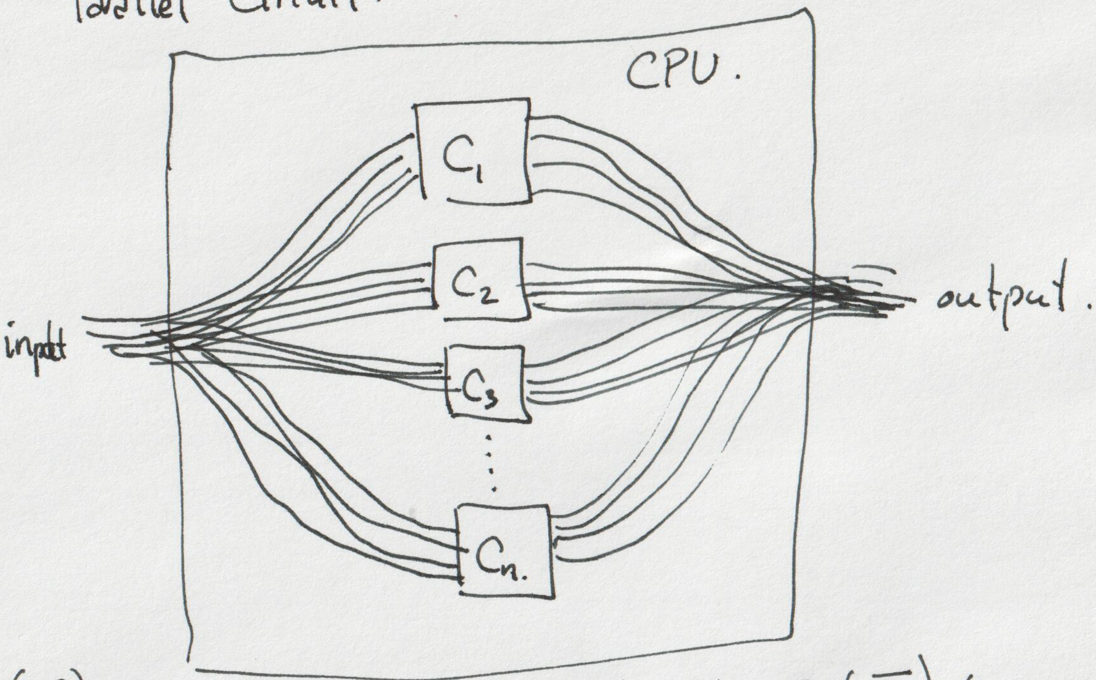
The events $E_1, E_2, \ldots, E_n$ are mutually independent..

$Pr(E_i) = p$ for each $i \in \{1, \ldots n\}$.

Example: $p = \frac{1}{2}$ $n = 10$.

$Pr(A) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$.

Parallel Circuit.



Example: $p = \frac{1}{2}$ $n = 10$.

$Pr(B) = \frac{1023}{1024}$.

$Pr(B)$ = "this CPU is still ~~useable~~ sellable."

$= Pr(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n)$

$Pr(B) = 1 - Pr(\overline{B})$.

$= 1 - (1-p)^n$.

$Pr(\overline{B})$ = "this CPU is worthless"

$= Pr(\overline{E}_1 \cap \overline{E}_2 \cap \overline{E}_3 \cap \cdots \cap \overline{E}_n)$

$= Pr(\overline{E}_1) \cdot Pr(\overline{E}_2) \cdot Pr(\overline{E}_3) \cdots Pr(\overline{E}_n)$

$= (1 - Pr(E_1))(1 - Pr(E_2)) \cdots (1 - Pr(E_n))$

$= \underbrace{(1-p)(1-p)\cdots(1-p)}_{n.}$

$= (1-p)^n$.

Data stream $x_1 \ldots\ldots x_n$.
- you don't know n.
- Return x_i with probabability $\frac{1}{n}$, for each $i \in \{1 \ldots n\}$.

```
i = 1.
while s.hasMoreData().
    x_i = s.nextData().
    with probability 1/i:
        z = x_i
    i = i+1.
return z
```

if random_int(i) = 1

$$x_1\, x_2\, x_3 \cdots\cdots x_{i-1}\, x_i$$

$$1 = \frac{i+1}{i+1}$$

A_i = "we set $z = x_i$ in iteration i", for $i \in \{1 \ldots n\}$.

$Pr(A_i) = \frac{1}{i}$, $A_1\, A_2\, A_3 \cdots\cdots A_n$ are mutually independent.

<u>Claim:</u> For each $i \in \{1 \ldots n\}$. the probability that procedure returns x_i is exactly $1/n$.

Proof: E = "the procedure returns x_i" $= A_i \cap \overline{A}_{i+1} \cap \overline{A}_{i+2} \cap \overline{A}_{i+3} \cap \cdots \cap \overline{A}_{n-1} \cap \overline{A}_n$

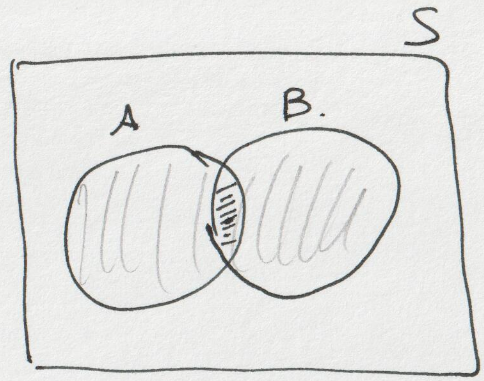
$$Pr(E) = Pr(A_i \cap \overline{A}_{i+1} \cap \overline{A}_{i+2} \cap \cdots \cap \overline{A}_{n-1} \cap \overline{A}_n).$$

$$= Pr(A_i) \cdot Pr(\overline{A}_{i+1}) \cdot Pr(\overline{A}_{i+2}) \cdots Pr(\overline{A}_{n-1}) \cdot Pr(\overline{A}_n)$$

$$= \frac{1}{i}\left(1 - \frac{1}{i+1}\right)\cdot\left(1 - \frac{1}{i+2}\right)\cdots\cdots\left(1 - \frac{1}{n-1}\right)\left(1 - \frac{1}{n}\right).$$

$$= \frac{1}{i}\left(\frac{i+1-1}{i+1}\right)\cdot\left(\frac{i+2-1}{i+2}\right)\cdots\left(\frac{n-1-1}{n-1}\right)\left(\frac{n-1}{n}\right) = \frac{1}{n} \stackrel{?}{=} \frac{1}{n}$$

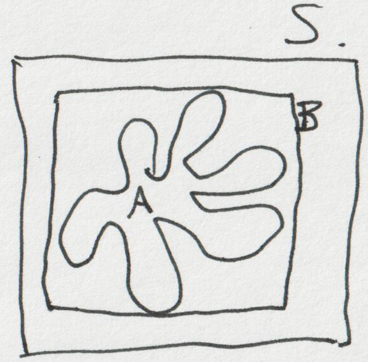
Defining events using logic.



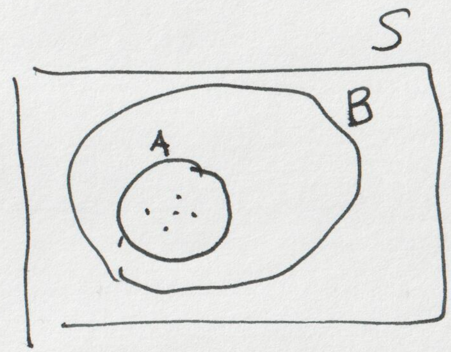
$A \overset{\vee}{\text{or}} B \iff A \cup B$

$A \text{ and } B \iff A \cap B$

$A \overset{\Rightarrow}{\text{implies}} B \iff A \subseteq B.$



$$\Pr(A) \leq \Pr(B).$$



$\text{not } A \iff \overline{A} = S \setminus A$

$\neg A$

If $A_1 \ldots\ldots A_n$ are mutually independent then.

$$\Pr(A_1 \cap \cdots\cdots \cap A_n) = \Pr(A_1) \cdot \Pr(A_2) \cdots\cdots \Pr(A_n).$$

$$\Pr(A_1 \cup A_2 \cup \cdots\cdots \cup A_n) = 1 - \Pr(\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n).$$

$$= 1 - \Pr(\overline{A}_1) \cdot \Pr(\overline{A}_2) \cdots\cdots \Pr(\overline{A}_n).$$

$$= 1 - (1 - \Pr(A_1))(1 - \Pr(A_2)) \cdots\cdots (1 - \Pr(A_n)).$$