Lemma: If $X : S \to \{0,1,2,3,4,\dots\} = \mathbb{N}$ is a non-negative integer random variable, then $E(X) = \sum_{i=1}^{\infty} Pr(X \geq i)$

Proof: Let $p_i = Pr(X=i)$, for all $i \in \mathbb{N}$.

$$Pr(X \geq i) = p_i + p_{i+1} + p_{i+2} + \cdots\cdots = \sum_{j=i}^{\infty} p_j$$

$$\begin{aligned}
\sum_{i=1}^{\infty} Pr(X \geq i) = \; & p_1 + p_2 + p_3 + p_4 + \cdots\cdots && [Pr(X \geq 1)] \\
& + p_2 + p_3 + p_4 + \cdots\cdots && [Pr(X \geq 2)] \\
& + p_3 + p_4 + \cdots\cdots && [Pr(X \geq 3)]. \\
& \qquad \ddots \\
= \; & p_1 + 2p_2 + 3p_3 + 4p_4 + \cdots\cdots \\
= \; & \sum_{i=1}^{\infty} i \cdot p_i = \sum_{i=1}^{\infty} i \cdot Pr(X=i) = E(X)
\end{aligned}$$

Reminders.

$\sum_{i=0}^{N} x^i = \frac{1-x^{N+1}}{1-x}$ $(x \neq 1)$

$\lim_{N \to \infty}$

$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ $(|x|<1)$

derivative

$\sum_{i=0}^{\infty} i \cdot x^{i-1} = \frac{1}{(1-x)^2}$

$H_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} 1/i$

$\int_1^{n} 1/x \, dx$

$\ln n \leq H_n \leq 1 + \ln n$

For non-negative integer X

$E(X) = \sum_{i=1}^{\infty} Pr(X \geq i)$.

Geometric Random Variable. geometric(p)

- Biased coin $Pr(H)=p$ $Pr(T)=1-p$.
- Flip the coin until the first time you get heads.
- G = the number of coin tosses until the first head.

$$E(G)=\sum_{i=1}^{\infty} Pr(G\geq i) = \sum_{i=1}^{\infty}(1-p)^{i-1} = \sum_{i=0}^{\infty}(1-p)^{i} = \frac{1}{1-(1-p)} = \frac{1}{p}. \quad \stackrel{?}{=} \frac{1}{p}.$$

probability that first $i-1$ tosses were tails.

$$E(G)=\sum_{i=1}^{\infty} i\cdot Pr(G=i) = \sum_{i=1}^{\infty} i\cdot(1-p)^{i-1}\cdot p = p\cdot\sum_{i=1}^{\infty} i\cdot(1-p)^{i-1} = p\cdot\frac{1}{(1-(1-p))^2} = p\cdot\frac{1}{p^2} = \frac{1}{p}.$$

Coupon Collector's Problem.

- An infinite sequence of coupons $C_1 C_2 C_3 \dots \dots$, each $C_i \in \{1, \dots, n\}$ uniformly random.

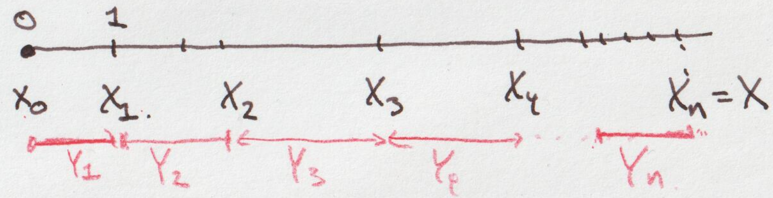
$$X = \min\{i : \{C_1, C_2, \dots, C_i\} = \{1, 2, \dots n\}\}.$$

$$X : S \to \{n, n+1, n+2, \dots\}.$$

$E(X)$

For each $k \in \{0, \dots n\}$, let $X_k = \min\{i : |\{C_1 \dots C_i\}| = k\}$. $\quad X = X_n$

For each $i \in \{1, \dots n\}$, define $Y_i = X_i - X_{i-1}$



$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} (X_i - X_{i-1}) = \cancel{X_1} - X_0 + \cancel{X_2} - \cancel{X_1} + \cancel{X_3} - \cancel{X_2} \cdots X_n - \cancel{X_{n-1}}$$

$$= X_n - X_0 = X_n = X$$

Y_i = "I have $i-1$ different coupons, and I'm waiting for a new one. How long?"

$$\frac{i-1}{n} \qquad \frac{n-i+1}{n} = p.$$

$$E(X) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \sum_{i=1}^{n} \frac{1}{n-i+1}$$

$$= n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{1}\right) = n \cdot H_n.$$

Y_i is a geometric $\frac{n-i+1}{n}$ rand. var.

Binomial Random Variables.

binomial(n,p)

- Biased coin $Pr(H) = p$.
- toss the coin n times
- B = the number of heads.

$$B : S \to \{0,1,2,\ldots\ldots n\}.$$

$$Pr(B=i) = \sum_{\{\omega \in S : B(\omega)=i\}} Pr(\omega) = \sum_{\{\omega \in S : B(\omega)=i\}} (1-p)^{n-i} p^i = \binom{n}{i} \cdot (1-p)^{n-i} p^i$$

$(1-p)(1-p)p(1-p)(1-p)\cdot p \qquad (1-p)\cdot$

$$\underbrace{T\,T\,H\,T\,T\,H\,T\,T\,H\,H\,T\cdots\cdots H}_{i \text{ heads.}}$$

$n-i$ tails.

$$(1-p)^{n-i} \cdot p^i$$

$$E(B) = \sum_{i=0}^{n} i \cdot Pr(B=i) = \sum_{i=0}^{n} i \cdot \binom{n}{i} (1-p)^{n-i} p^i \overset{?}{=} p \cdot n.$$

Let. $B_i = 1$ if i^{th} coin toss is heads and 0 otherwise.

$$E(B) = E\left(\sum_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} E(B_i) = \sum_{i=1}^{n} p = p \cdot n.$$

$$B_i = \begin{cases} 1 & \text{if } i^{th} \text{ coin toss is heads} \\ 0 & \text{otherwise.} \end{cases}$$

$$E(B_i) = Pr(B_i = 1) = p.$$