Reminder: (S, Pr)

- For events $A, B \subseteq S$ with $Pr(B) > 0$, $Pr(A|B) = \dfrac{Pr(A \cap B)}{Pr(B)}$.

Definition: Two events $A, B \subseteq S$ are independent if and only if $Pr(A \cap B) = Pr(A) \cdot Pr(B)$.

Notice: If A and B are independent and $Pr(B) > 0$ then.

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{Pr(A) \cdot Pr(B)}{Pr(B)} = Pr(A).$$

Definition: If $Pr(A \cap B) \neq Pr(A) \cdot Pr(B)$ then A and B are not independent.

Example: Roll two dice. $S = \{(d_1, d_2) : d_1, d_2 \in \{1,2,3,4,5,6\}\}$ $|S| = 36$. $Pr(\omega) = \frac{1}{|S|} = \frac{1}{36}$.

$A =$ "$d_1 = 6$" $= \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$. $Pr(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$.

$B =$ "$d_2 = 4$" $= \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$. $Pr(B) = \frac{|B|}{|S|} = \frac{6}{36} = \frac{1}{6}$.

$A \cap B =$ "$d_1 = 6, d_2 = 4$" $= \{(6,4)\}$. $Pr(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = Pr(A) \cdot Pr(B)$.

$\therefore$ A and B are independent.

$A = \text{"}d_1 + d_2 = 7\text{"} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ $\quad Pr(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}.$

$B = \text{"}d_2 = 4\text{"} = \{(1,4),(2,4),\ldots(6,4)\}.$ $\quad Pr(B) = \frac{1}{6}.$

$A \cap B = \{(3,4)\}.$ $\quad Pr(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = Pr(A) \cdot Pr(B).$

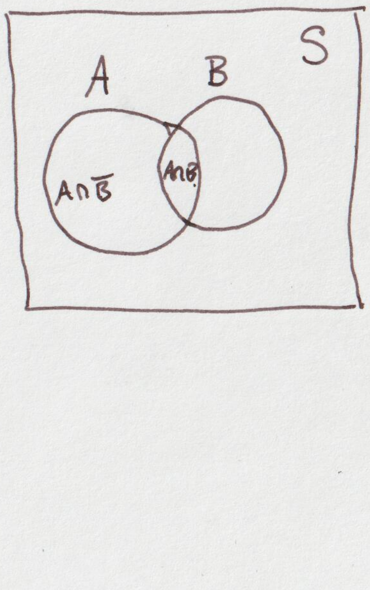
$A = \text{"}d_1 + d_2 = 5\text{"} = \{(1,4),(2,3),(3,2),(4,1)\}$ $\quad Pr(A) = \frac{|A|}{|S|} = \frac{4}{36} = \frac{1}{9}.$

$B = \text{"}d_2 = 4\text{"}$ $\quad Pr(B) = \frac{1}{6}.$

$A \cap B = \{(1,4)\}.$ $\quad Pr(A \cap B) = \frac{1}{36}. \neq \frac{1}{9} \cdot \frac{1}{6} = Pr(A) \cdot Pr(B).$

$\therefore$ A and B are <u>not independent</u>.

<u>Claim</u>: If A and B are independent then A and $\overline{B}$ are independent.



<u>Proof</u>: Need to show that $Pr(A \cap \overline{B}) = Pr(A) \cdot Pr(\overline{B}) = Pr(A) \cdot (1 - Pr(B)).$

$$\begin{aligned} Pr(A) &= Pr(A \cap B) + Pr(A \cap \overline{B}) \\ &= Pr(A) \cdot Pr(B) + Pr(A \cap \overline{B}). \end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned} Pr(A \cap \overline{B}) &= Pr(A) - Pr(A) \cdot Pr(B). \\ &= Pr(A)(1 - Pr(B)) \\ &= Pr(A) \cdot Pr(\overline{B}). \end{aligned}$$

<u>Definition:</u> A sequence of events $A_1 \ldots A_n$ is <u>pairwise independent</u> if for each $1 \le i < j \le n$.

$$Pr(A_i \cap A_j) = Pr(A_i) \cdot Pr(A_j).$$

$\binom{n}{2}$ things to check.

weak!

<u>Definition:</u> $A_1 \ldots A_n$ are <u>mutually independent</u> if for every $k \in \{2, \ldots n\}$ and every

$$1 \le i_1 < i_2 < i_3 < \cdots < i_k \le n$$

$2^n - n - 1$

$$Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = Pr(A_{i_1}) \cdot Pr(A_{i_2}) \cdots Pr(A_{i_k}).$$

great!

Example: Random bitstring of length 2, $b_1 b_2$.

$$S = \{00, 01, 10, 11\}.$$

$A = $ "$b_1 = 1$" $= \{10, 11\}$. $\quad Pr(A) = \frac{|A|}{|S|} = \frac{2}{4} = \frac{1}{2}$.

$B = $ "$b_2 = 1$" $= \{01, 11\}$. $\quad Pr(B) = \frac{1}{2}$

$C = $ "$(b_1 + b_2) \bmod 2 = 1$" $\quad Pr(C) = \frac{1}{2}$

$= $ "$b_1 \ne b_2$".

$= \{01, 10\}$.

$$A \cap B \cap C = \emptyset$$

$Pr(A \cap B) = Pr(\{11\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = Pr(A) \cdot Pr(B)$ ✓

$Pr(B \cap C) = Pr(\{01\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = Pr(B) \cdot Pr(C)$. ✓

$Pr(A \cap C) = Pr(\{10\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = Pr(A) \cdot Pr(C)$ ✓

$\therefore$ A, B, C are pairwise independent.

$Pr(A \cap B \cap C) = 0 \ne \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = Pr(A) \cdot Pr(B) \cdot Pr(C)$ ✗.

$\therefore$ A, B, C are not mutually independent.

Annie, Boris, and Charlie write an exam.

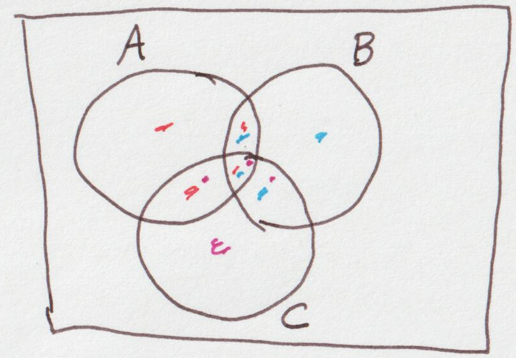
$S=\{(a,b,c) : a,b,c \in \{UNS, SAT\}\}$. $|S|=8$.

- 1 question exam.

A = "Annie answers correctly" $\Pr(A) = 0.9 = \frac{9}{10}$

B = "Boris answers correctly" $\Pr(B) = 0.9 = \frac{9}{10}$.

C = "Charlie answers correctly" $\Pr(C) = 0.6 = \frac{6}{10}$



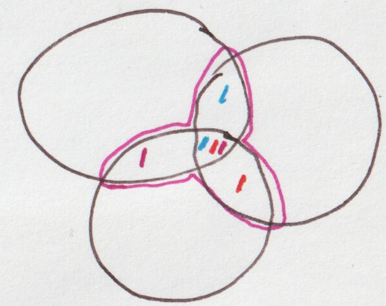
Scenario 1: A, B, and C are mutually independent.

E = "at least 2 students answer correctly" $= (A\cap B)\cup(B\cap C)\cup(A\cap C)$.

$$\Pr(E) = \Pr(A\cap B) + \Pr(B\cap C) + \Pr(A\cap C) - 2\Pr(A\cap B\cap C).$$

$$= \Pr(A)\cdot\Pr(B) + \Pr(B)\cdot\Pr(C) + \Pr(A)\cdot\Pr(C) - 2\cdot\Pr(A)\cdot\Pr(B)\cdot\Pr(C).$$

$$= \frac{9}{10}\cdot\frac{9}{10} + \frac{9}{10}\cdot\frac{6}{10} + \frac{9}{10}\cdot\frac{6}{10} - 2\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{6}{10} = \frac{459}{500} = 0.918$$



Scenario: ~~A~~ Charlie copies Annie's Answer.

E = "A

$$\Pr(E) = \Pr(A) = \frac{9}{10} = 0.9$$