Indicator Random Variables. (S, Pr) = a probability space.

- Any function $B : S \to \{0, 1\}$ is called an <u>indicator random variable.</u>

- Fact: For any indicator rand. var. B, $E(B) = Pr(B=1)$.

$$E(B) = Pr(B=1)\cdot 1 + Pr(B=0)\cdot 0$$
$$= Pr(B=1).$$

Example: Random bitstring $b_1 \ldots b_n$. $S = \{0,1\}^n$ $Pr(\omega) = \frac{1}{|S|} = \frac{1}{2^n}$

$X(b_1 b_2 \ldots b_n)$ = "the number of 1's in $b_1 \ldots b_n$" $= \sum_{i=1}^{n} b_i$ $\quad [X : S \to \{0, 1, 2, \ldots n\}.]$

$$\boxed{E(X) = \sum_{i=0}^{n} Pr(X=i)\cdot i = \underbrace{\sum_{i=0}^{n} \frac{\binom{n}{i}}{2^n}\cdot i}_{} \overset{?}{=} \frac{n}{2}.}$$

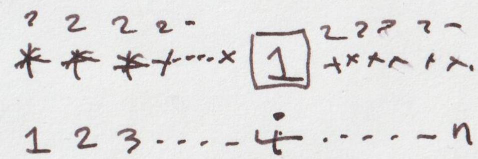
Hard way.

"$X = i$" = "$b_1 \ldots b_n$ has exactly i 1's"

$$Pr(X=i) = \frac{|\text{"}X=i\text{"}|}{2^n} = \frac{\binom{n}{i}}{2^n}.$$

For each $i \in \{1 \ldots n\}$, define $B_i : S \to \{0, 1\}$.

$$B_i(b_1 \ldots b_n) = b_i$$



$$E(X) = \underbrace{E\left(\sum_{i=1}^{n} B_i\right)}_{\text{expected value of a sum.}} = \sum_{i=1}^{n} E(B_i) = \sum_{i=1}^{n} \frac{1}{2}$$
$$= \frac{n}{2}.$$

$$E(B_i) = Pr(B_i = 1)$$
$$= Pr(b_i = 1).$$
$$= \frac{|\text{"}b_i = 1\text{"}|}{2^n}$$
$$= \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

<u>Lin. of Expectation:</u>

For any rand. variables $X_1 \ldots X_n$

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

$$\boxed{\begin{matrix} 2^{i}-1 \\ 2^{i} \end{matrix}}$$

Def'n: $b_1 \ldots b_n$ is a bitstring.

- A run of length k starts at i if $b_i = b_{i+1} = b_{i+2} = \cdots = b_{i+k-1}$. [for $i \le n-k+1$].

$n = 1024$

$k = 11$.

$$\frac{1024}{2^{10}} = 1$$

$X(b_1 \ldots b_n)$ = the number of runs of length k in $b_1 \ldots b_n$.

$$E(X) = E\left(\sum_{i=1}^{n-k+1} X_i\right)$$

For $i \in \{1 \ldots n-k+1\}$, $X_i = \begin{cases} 1 & b_i = b_{i+1} = \cdots = b_{i+k-1} \\ 0 & \text{otherwise.} \end{cases}$

$$= \sum_{i=1}^{n-k+1} E(X_i) = \sum_{i=1}^{n-k+1} \Pr(X_i = 1) = \sum_{i=1}^{n-k+1} \frac{1}{2^{k-1}} = \frac{(n-k+1)}{2^{k-1}}.$$

$$1 \; 2 \; 3 \; 4 \cdots \underbrace{i \cdots i+k-1}_{k.}$$

$$|\text{"}X_i = 1\text{"}| = 2^{n-k} \cdot 2 = 2^{n-k+1}.$$

$$\Pr(X_i = 1) = \frac{2^{n-k+1}}{2^n} = \frac{1}{2^{k-1}}$$

<u>Birthday paradox</u> n people, d days in a year.

$$S = \{d_1 \dots d_n : d_i \in \{1 \dots d\}\}$$

$X =$ "the number of pairs of people that have the same birthday"

For each i,j with $1 \le i < j \le n$ $X_{i,j} = \begin{cases} 1 & \text{if } d_i = d_j \\ 0 & \text{otherwise.} \end{cases}$

$$E(X) = E\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}\right) = \sum\sum E(X_{i,j}) = \sum\sum \Pr(X_{i,j} = 1) = \sum\sum \frac{1}{d} = \frac{\binom{n}{2}}{d} = \frac{n(n-1)}{2d}$$

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InsertionSort(a, n)    // a_1 .... a_n is an array
1  for i = 1 to n:
2      j = i
3      while j > 1 and a_{j-1} > a_j:
4          swap a_{j-1} <-> a_j   ] = ??
5          j = j - 1.             ]
```

2 4 3 1

2 3 4 1

2 3 1 4

2 1 3 4

1 2 3 4

Suppose $a_1 \dots a_n$ is a random permutation of $1, \dots, n$.

$|S| = n!$ $\quad$ $Pr(\omega) = \frac{1}{n!}$ for each $\omega \in S$.

$X(a_1 \dots a_n) =$ "the number of times line 4 executes" when sorting $a_1 \dots a_n$

For $i < j$

$$X_{i,j}(a_1 \dots a_n) = \begin{cases} 1 & \text{if } j \text{ appears before } i \text{ in } a_1 \dots a_n. \\ 0 & \text{otherwise.} \end{cases}$$

$$E(X) = E\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}\right) = \sum\sum E(X_{i,j}) = \sum\sum Pr(X_{i,j} = 1) = \sum\sum \frac{(n!)/2}{n!} = \sum\sum \frac{1}{2}$$

$$= \binom{n}{2} / 2.$$

FindMax(a, n) // $a_1 \ldots a_n$ is an array of numbers.

- max = $-\infty$
- for i = 1 to n
  - if $a_i >$ max

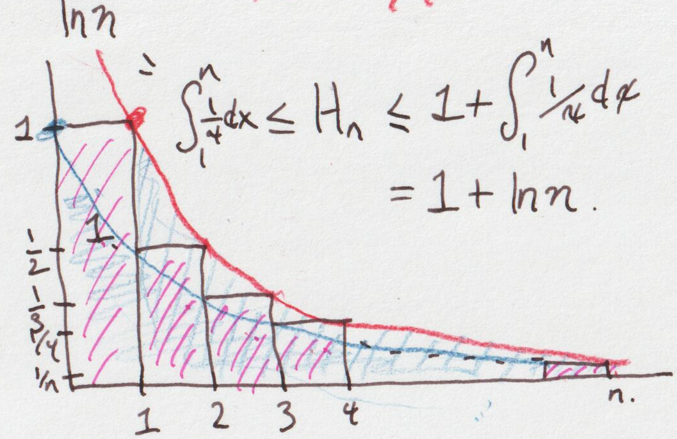
    max = a_i ?? (*).

X = "the number of times line (*) executes".

Suppose $a_1 \ldots a_n$ is a random permutation of $1 \ldots n$.

$|S| = n!$ $\quad Pr(\omega) = \frac{1}{n!}$, for each $\omega \in S$.

$f(x) = 1/x$.



$$\int_1^n \frac{1}{x}dx \le H_n \le 1 + \int_1^n \frac{1}{x}dx$$

$$= 1 + \ln n.$$

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

$$= \sum_{i=1}^{n} Pr(X_i = 1) = \sum \frac{(n!)/i}{n!} = \sum \frac{1}{i}$$

$$= \sum_{i=1}^{n} 1/i \overset{\text{def}}{=} H_n$$

"n^{th} harmonic number"

For each $i \in \{1 \ldots n\}$

$$X_i = \begin{cases} 1 & \text{if line (*) executes on iteration } i. \\ 0 & \text{otherwise.} \end{cases}$$

$$X_i = \begin{cases} 1 & \text{if } a_i = \max\{a_1 \ldots a_i\}. \\ 0 & \text{otherwise.} \end{cases}$$

$$\binom{n}{i} \cdot (i-1)!\,(n-i)!$$

$$= \frac{n!}{i!\,\cancel{(n-i)}!} \cdot (i-1)! \cdot \cancel{(n-i)}! = \frac{n!}{i}$$

1. choose the values to put in $a_1 \ldots a_i$
2. choose order of $a_1 \ldots a_i$