d d d d d

1 . . . n.

COMP2804 Midterm Exam, Fall 2023

a, b, c, d, $4^n = \binom{n}{k} \cdot 3^{n-k}$ $\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} 3^{n-k}$

$= \sum_{k=0}^{n} \binom{n}{k} \cdot 3^k$

$= 4^n$

This a closed book exam with a duration of 1h20m. No computers, calculators, phones, or other aids are allowed.

This paper has wide margins. Feel free to write in them. You **may** take this paper with you when you are done.

This is a multiple-choice Scantron exam. Be sure to complete your name and student number and answer all questions on the Scantron sheet provided to you.

Select exactly one option for each question. In cases where you believe there is more than one correct option, select the most accurate option.

Marking scheme: Each of the 17 questions is worth 1 mark.

Reminders:

- Binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Newton's Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- Fibonacci numbers:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

0
1
1
2
3

Question 1. Pat's Pub has just introduced the *Super Value Meal* that includes one appetizer (fries, onion rings, or mozzarella sticks), one entrée (hamburger, grilled cheese, tuna melt, or steak melt), one desert (apple pie or pecan pie). The meal also comes with any selection of three drinks (red beer, brown beer, pale beer, black beer, or water).

How many ways does a customer at Pat's Pub have to choose their meal.

(a) $3 + 4 + 2 + 5$
(b) $3 \cdot 4 \cdot 2 \cdot 5$ ×
(c) $3 \cdot 4 \cdot 2 \cdot \binom{5}{3}$
(d) $3 \cdot 4 \cdot 2 \cdot 3^5$ ×
or (e) $3 \cdot 4 \cdot 2 \cdot 5^3$

$3 \cdot 4 \cdot 2$

Question 2. *Edison*, the new electric car company has five models of cars, each of which is available in the colours red, blue, green, yellow, cyan, or magenta, and each model has three different battery options.

1 2 3 4 5 6

Edison customers get to choose *up to* two from a set of five available add-on features: heated seats, partial self-driving, fancy rims, gold steering wheel, or cosmetic exhaust pipe.

How many ways are there for a customer to choose their Edison car?

(a) $5 + 6 + 3 + 2 + 5$
(b) $5 \cdot 6 \cdot 3 \cdot 2 \cdot 5$
×(c) $5 \cdot 6 \cdot 3 \cdot \binom{5}{2}$
×(d) $5 \cdot 6 \cdot 3 \cdot 2 \cdot \binom{5}{2}$
(e) $5 \cdot 6 \cdot 3 \cdot (6 + \binom{5}{2})$

$5 \cdot 6 \cdot 3 \left(1 + \binom{5}{1} + \binom{5}{2}\right)$

$1+5$

$6.$

Question 3. The School of Computer Science is taking a group photo of all 44 professors (using an ultra-wide-angle lens), standing side-by-side in a single line. Four of the professors (Jit, Anil, Pat, and Michiel) love each other and want to be standing consecutively in the line.

How many ways are there to line up the professors so that Jit, Anil, Pat, and Michiel are together?

(a) $43! \cdot 42$
(b) $41!$
(c) $41! \cdot 4!$
(d) $41! \cdot \binom{41}{4}$
(e) $41! \cdot 4$

JAPH + 40 professors.

$41! \cdot 4!$

Question 4. Continuing the previous question, two of the professors (Greg and Louis) can't stand each other and will fight and ruin the photo if they are standing side-by-side.

How many ways are there to line up the professors so that Jit, Anil, Pat, and Michiel are together and Greg and Louis have at least one other professor between them?

(a) $44!$
(b) $40! \cdot 4! \cdot 41$
(c) $40! \cdot 4! \cdot 38$
(d) $40! \cdot 4! \cdot 39$
(e) $2 \cdot 40! \cdot 4!$

JAPM   G 39 professors $40! \cdot 4! \cdot 39$.

1 2 3 … 40

L

Question 5. I'm going to the animal shelter to adopt some cats. There are c cats at the shelter and I will adopt 1, 2, or 3 of them. The person working at the shelter informs me that each cat needs to have its own distinct litter box, which I can buy there. There are 10 different colours of litter boxes and each cat needs a box of a different colour. I need to choose my (1, 2, or 3) cats and choose (1, 2, or 3) differently coloured litter boxes. How many options do I have?

1 or 2 or 3.

(a) $3 \cdot c \cdot \binom{10}{3}$
(b) $3 \cdot \binom{c}{3} \cdot \binom{10}{3}$
(c) $10c + \binom{c}{2} \cdot \binom{10}{2} + \binom{c}{3} \cdot \binom{10}{3}$ ✓
(d) $\binom{10c}{1} + \binom{10c}{2} + \binom{10c}{3}$
(e) $10c \cdot 9(c-1) \cdot 8(c-2)$

$c \cdot 10 + \binom{c}{2}\cdot\binom{10}{2} + \binom{c}{3}\cdot\binom{10}{3}$.

Question 6. How many strings can be obtained by rearranging the letters of the word

1 2 3 4 5 6 7 8 9 10 11 12 13.

CHATTAHOOCHEE

(a) $13!$
(b) $13!/(3 \cdot 2^6)$?
(c) $\binom{13}{2} \cdot \binom{11}{2} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot \binom{3}{2}$ = ?
(d) $\binom{13}{3} \cdot \binom{10}{3} \cdot \binom{7}{3} \cdot \binom{4}{3}$
(e) $\binom{13}{3} \cdot \binom{10}{3} \cdot \binom{7}{3} \cdot \binom{4}{3}$

2×C   13
3×H
2×A
2×T
2×O
2×E

$= \binom{13}{2}\binom{11}{3}\cdot\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$

A O C E A E H T H C T H O
1 2 3 4 5 6 7 8 9 10 11 12 13.

$= \frac{13\cdot 12}{2} \cdot \frac{11\cdot 10\cdot 9}{3!} \cdot \frac{8\cdot 7}{2} \cdot \frac{6\cdot 5}{2} \cdot \frac{4\cdot 3}{2} \cdot \frac{2\cdot 1}{2} = \frac{13!}{3!\, 2^5} = \frac{13!}{3\cdot 2^6}$

$3! = 3\cdot 2\cdot 1$

$4^n = (3+1)^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} \cdot 7$ ddd

$= 4^n$. ...d d d

Question 7. What does $\boxed{\sum_{k=0}^{n} \binom{n}{k} 3^k}$ count?

$\binom{n}{3} 3^{(n-3)}$ $\binom{n}{k} 3^{n-k}$

(a) The number of strings over the alphabet $\{a, b, c\}$.
(b) The number of strings over the alphabet $\{a, b, c\}$ that contain at least one a.
(c) The number of strings over the alphabet $\{a, b, c, d\}$.
(d) The number of strings over the alphabet $\{a, b, c, d\}$ that contain at least one a. $\binom{n}{k} = \binom{n}{n-k}$.
(e) The number of strings over the alphabet $\{a, b, c, d\}$, minus one.

Question 8. How many ways are there to partition a set of size 4 into two sets, each of size 2? (Note: We consider $\{\{A, B\}, \{C, D\}\}$ and $\{\{C, D\}, \{A, B\}\}$ to be the same partition. Since Halloween is close, think of putting four different pieces of candy into two identical paper bags.) $\binom{n}{n-k} 3^{n-k}$ $\sum \binom{n}{k} \cdot 3^k$

(a) $\binom{4}{2}$
(b) 2^4
(c) $\binom{4}{2}/3$
(d) 12
(e) 3

$\binom{4}{2}/2$ $\binom{4}{2}/2$ CD AB $3 = 2n-1$

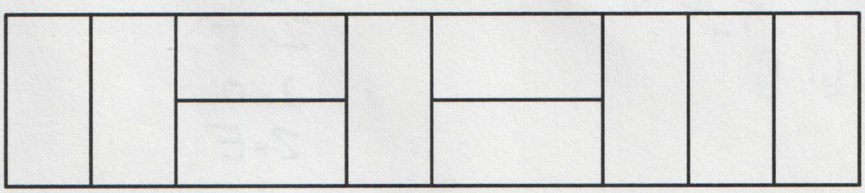
$\frac{4 \cdot 3}{2}/2 = 3$. AB CD AB CD 3

Question 9. Which of the following is true, for any positive integer n? (Hint: This is related to the previous question.)

$4 = 2n$. $\binom{3}{1}$.

(a) $\binom{2n}{n} = 2 \cdot \binom{n}{n/2}$
(b) $\binom{2n-1}{n-1} = \binom{2n}{n}/2$ $\binom{2n}{n}/2$. $=$ $\binom{2n-1}{n-1}$ A n-1
(c) $\binom{2n}{n-1} = \binom{2n}{n/2}$
(d) $\binom{2n}{n-1} = 2 \cdot \binom{2n}{n/2}$
(e) $\binom{2n}{n-1} = \binom{2n}{n/2}^4$

Question 10. How many ways are there to tile a $2 \times n$ rectangle using tiles of side length 1×2. For example, here is a tiling of the 2×10 rectangle.



2 1

(a) 2^n
(b) f_{n-1}
(c) f_n
(d) f_{n+1}
(e) f_{n+2}

$\boxed{F_n = F_{n-1} + F_{n-2}}$

$F_0 = 1$

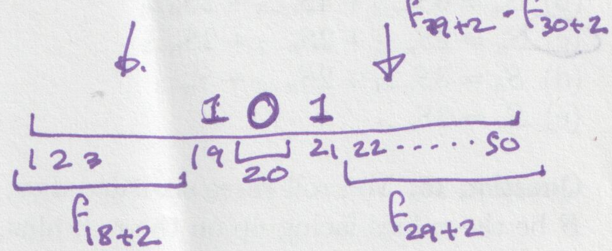
$F_1 = 1$ n-1 n-2.

$F_2 = 2$.

3.

Question 11. A bitstring is called 00-free if it does not contain two consecutive zeros. In class, we have seen that, for any integer $m \geq 1$ the number of 00-free bitstrings of length m is equal to the $(m+2)$th Fibonacci number f_{m+2}.

What is the number of 00-free bitstrings of length 50 that have a 0 at position 20? More precisely, how many 00-free bitstrings are there of the form $b_1, \ldots, b_{19}, 0, b_{21}, \ldots, b_{50}$?



(a) $f_{21} + f_{32}$
(b) $f_{19} + f_{30}$
(c) $f_{21} \cdot f_{32}$
(d) $f_{20} \cdot f_{31}$
(e) None of the other answers is correct

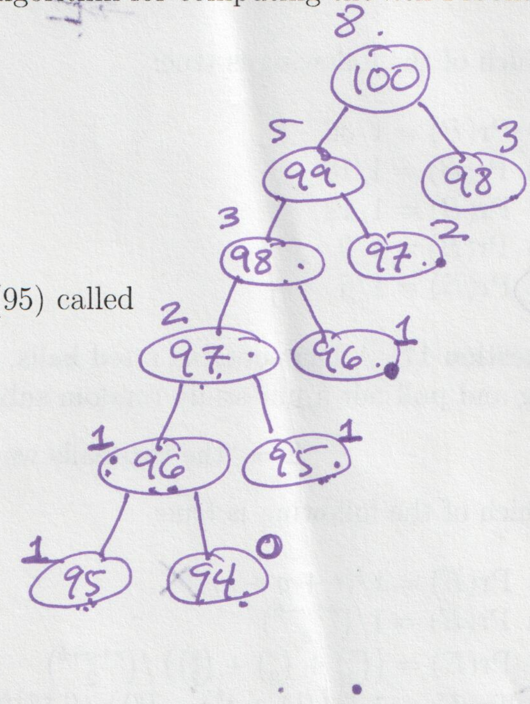
Question 12. We have n *distinct* soldiers $s_1, \ldots, s_n$ and we want to partition them into *distinct* groups $g_1, \ldots, g_k$, where each group contains either 3 soldiers or 2 soldiers. (The number of groups, k, can change depending on how we do our partition). For any $n \geq 0$, let T_n denote the number of ways of doing this. Then $T_0 = 1$, $T_1 = 0$, and $T_2 = 1$. Which of the following is true, for any $n \geq 3$?

$\binom{n}{2} T_{n-2} + \binom{n}{3} \cdot T_{n-3}.$

(a) $T_n = f_{\lfloor n/3 \rfloor}$.
(b) $T_n = f_{\lfloor n/2 \rfloor}$.
(c) $T_n = T_{n-3} + T_{n-2}$.
(d) $T_n = \binom{n}{3} \cdot T_{n-3} + \binom{n}{2} \cdot T_{n-2}$.
(e) $T_n = 2^{n/2.5}$

Question 13. Consider the following (very slow) algorithm for computing the nth Fibonacci number f_n:

```
FIB(n)
  if n ≤ 1 then
    return n
  return FIB(n − 1) + FIB(n − 2)
```



When running FIB(100), how many times is FIB(95) called

(a) 2
(b) 3
(c) 5
(d) 8
(e) 13

Question 14. You an infinite supply of identical red marbles, an infinite supply of identical blue marbles, and an infinite supply of identical green marbles. Let S_n be the number of ways of placing n of these marbles on a line so that you never have four consecutive marbles of the same colour. Which of the following is true, for any $n \geq 4$?

(a) $S_n = 6S_{n-2} + 6S_{n-3} + 6S_{n-4}$
(b) $S_n = 6S_{n-2} + 4S_{n-3} + 2S_{n-4}$
(c) $S_n = 2S_{n-1} + 2S_{n-2} + 2S_{n-3}$
(d) $S_n = 3S_{n-1} + 2S_{n-2} + S_{n-3}$
(e) $S_n = 3S_{n-1}$

b. 6.

S_{n-1}

oo S_{n-2}

ooo S_{n-3}

Question 15. You roll three six-sided dice, one red, one green, and one blue. Let R, G, and B be the values facing up on the red, blue, and green die, respectively. Define the event:

$$A = \text{“}R = G\text{”} \ .$$

Which of the following is true:

(a) $\Pr(A) = 1/36$
(b) $\Pr(A) = 1/18$
(c) $\Pr(A) = 1/12$
(d) $\Pr(A) = 1/9$
(e) $\Pr(A) = 1/6$

$S = \{(r,g,b) : r,g,b \in \{1,\ldots,6\}\}$.
$|S| = 6^3$
$A = \{(x,x,b) : x,b \in \{1..6\}\}$ $|A| = 6^2$
$\Pr(A) = \frac{|A|}{|S|} = \frac{6^2}{6^3} = \frac{1}{6}$.

Question 16. You roll three six-sided dice, one red, one green, and one blue. Let R, G, and B be the values facing up on the red, blue, and green die, respectively. Define the event:

$$B = \text{“}R + G = 7\text{”} \ .$$

Which of the following is true:

(a) $\Pr(B) = 1/36$
(b) $\Pr(B) = 1/18$
(c) $\Pr(B) = 1/12$
(d) $\Pr(B) = 1/9$
(e) $\Pr(B) = 1/6$

$E = \{(x, 7-x, b) : x,b \in \{1..6\}\}$.
$|E| = 6^2$
$\Pr(E) = \frac{|E|}{|S|} = \frac{6^2}{6^3} = \frac{1}{6}$.

Question 17. A bag contains r red balls, b blue balls, and g green balls. We reach into the bag and pull out a uniformly random subset of two balls. Define the event:

$$E = \text{“the two balls we chose have different colours”} \ .$$

Which of the following is true:

(a) $\Pr(E) = 1/(r + g + b)$
(b) $\Pr(E) = 1/\binom{r+g+b}{2}$
(c) $\Pr(E) = \left(\binom{r}{2} + \binom{b}{2} + \binom{g}{2}\right) / \binom{r+g+b}{2}$
(d) $\Pr(E) = 1 - \left(\binom{r}{2} + \binom{b}{2} + \binom{g}{2}\right) / \binom{r+g+b}{2}$
(e) $\Pr(E) = 1 - \binom{r}{2} \cdot \binom{b}{2} \cdot \binom{g}{2} / \binom{r+g+b}{2}$

$\rightarrow |S| = \binom{r+g+b}{2}$
$|\bar{E}| = \binom{r}{2} + \binom{g}{2} + \binom{b}{2}$

$\Pr(E) = 1 - \Pr(\bar{E})$.

$\Pr(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{\binom{r}{2}+\binom{g}{2}+\binom{b}{2}}{\binom{r+g+b}{2}}$