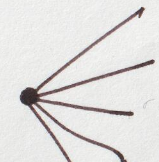
The Pigeonhole Principle.

- If $K+1$ pigeons fly into K holes, then some hole contains at least 2 pigeons
- If m pigeons fly into n holes, then some hole contains at least $\lceil \frac{m}{n} \rceil$ pigeons.

Eg. $\overset{m}{\overset{\shortparallel}{25}}$ pigeons into $n=12$ holes. $\lceil \frac{m}{n} \rceil = \lceil 2+\frac{1}{12} \rceil = 3$.



- Simon drinks at least 1 beer every day.
- Last April (which has 30 days), Simon drank exactly 45 beers.

$\lceil \frac{5}{2} \rceil = \lceil 2.5 \rceil = 3$.

- **Claim:** There was a sequence of consecutive days in April when Simon drank <u>exactly</u> 14 beers.

$\lceil \frac{45}{30} \rceil = \lceil 1.5 \rceil = 2$.

Proof: For each $i \in \{1,\dots,30\}$ let a_i be the number of beers Simon drank on April i.

(i) - $a_i \geq 1$ for each $i \in \{1,\dots,30\}$.

(ii) - $a_1+a_2+\cdots+a_{30} = 45$.

- For each $i \in \{1,\dots,30\}$ let $b_i = a_1+a_2+\cdots+a_i$

(i) - $b_{i+1} \geq b_i + 1$

(ii) $b_{30} = 45$

$\underbrace{b_1\, b_2\, \boxed{b_3}, \dots\dots, b_{30}, \boxed{b_1+14}, b_2+14, b_3+14, \dots, b_{30}+14}_{m=60}$

in the range $\{1,2,3,\dots\dots,59\}$. ($n = 59$)

some number must appear at least twice.

$b_i = b_j + 14 \qquad i > j \qquad 14 = b_i - b_j = a_{j+1}+a_{j+2}+\cdots+a_i$

$b_i = a_1+a_2+a_3+\cdots+a_j+a_{j+1}+\cdots+a_i$

$b_j = a_1+a_2+a_3+\cdots+a_j$

• Let S be an $(n+1)$-element subset of $\{1,2,3,\ldots\ldots,2n\}$.

• **Claim:** S contains two different numbers x and y such that x divides y $\left[\frac{y}{x} \text{ is an integer}\right]$.

• Let $S = a_1, a_2, a_3 \ldots a_{n+1}$.

$n=4$, $2n=8$

• For each $i \in \{1,\ldots n+1\}$, write a_i as $2^{k_i} q_i$ where $\underset{\text{integer}}{k_i} \geq 0$ and q_i is odd.

$\{8,4,5,6,7\}$.

• Define: $f: \overbrace{\{1,\ldots,n+1\}}^{m=n+1} \to \overbrace{\{1,3,5,7,9,\ldots,2n-1\}}^{\cancel{m=n}\ n}$.

$a_i = 40 = 2 \cdot 20 = 2^2 \cdot 10 = 2^3 \cdot 5$

$f(i) = q_i$

$q_1, q_2, q_3 \cdots q_{n+1}$

By PHP $f(i) = f(j)$ for some $i \neq j$.

$k_i = 3$ $q_i = 5$.

$a_i = 2^{k_i} \cdot q_i$

$k_i \neq k_j$

$a_j = 2^{k_j} \cdot q_j = 2^{k_j} \cdot q_i$

Assume without loss of generality that $k_i > k_j$

$$\frac{a_i}{a_j} = \frac{2^{k_i} \cdot \cancel{q_i}}{2^{k_j} \cdot \cancel{q_i}} = \underbrace{2^{k_i - k_j}}_{\text{integer}}$$

$k_i - k_j \geq$ integer

Erdős-Szekeres Theorem: Let S: $a_1, a_2, \dots, a_{n^2+1}$ be a sequence of n^2+1 distinct numbers. Then S contains an increasing subsequence of length $n+1$ or S contains a decreasing subsequence of length $n+1$.

$f(i) =$	1	4	4	3	3	2	2	1	2	1
	10,	5,	3,	7,	4,	8,	2,	9,	1,	6

$n = 3$.

- For each $i \in \{1, \dots, n^2+1\}$ let $f(i)$ be the length of the longest increasing subsequence that begins with a_i

10, 9, 8, 7, 6, ... 1

1, 2, 3, ... 10

- If $f(i) \geq n+1$ for some i then we're done.
- Otherwise $f(i) \in \{1, 2, 3, \dots, n\}$ for all $i \in \{1, \dots, n^2+1\}$

$$\left\lceil n + \frac{1}{n} \right\rceil = n+1$$

$$f : \{1, 2, \dots, n^2+1\} \to \{1, 2, 3, \dots, n\}.$$

By PHP, ~~there are~~ some number appears at least $\left\lceil \frac{n^2+1}{n} \right\rceil = n+1$. ~~times~~

Suppose $f(i_1) = f(i_2) = f(i_3) = \dots = f(i_{n+1})$.

$$\boxed{\cdot} > \underset{i_{n-1}}{\overset{k}{\boxed{\cdot}}} > \overset{k}{\boxed{a_{i_n}}} \quad \boxed{\cancel{a_{i_n}}} > \overset{k}{\boxed{a_{i_{n+1}}}}$$

$a_{i_1}, a_{i_2}, \dots, a_{i_{n+1}}$ is a decreasing subsequence.