The Product Rule

- If a procedure P has m steps and, for each $i \in \{1, \dots, m\}$ the number of ways to execute step i is N_i, then the number of ways to execute P is $N_1 \times N_2 \times N_3 \times \dots \times N_m$.

- If P generates elements in a set S and
  - (i) – For each $x \in S$, there is an execution of P that generates x; and (onto)
  - (ii) – Any two different executions of P generate different elements of S. (one-to-one)

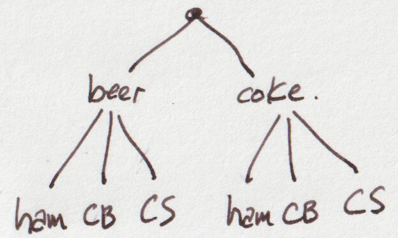
$|S| = N_1 \times N_2 \times N_3 \times \dots \times N_m$.

drinks = {beer, coke}.

foods = {hamburger, CB, CS}.

$N_1 = 2$.

$N_2 = 3$.



$S = \left\{ \begin{array}{l} (\text{beer}, \text{ham}), (\text{beer}, \text{CB}), (\text{beer}, \text{CS}) \\ (\text{coke}, \text{ham}), (\text{coke}, \text{CB}), (\text{coke}, \text{CS}) \end{array} \right\}$.

(beer, CS).

A bitstring is a sequence of 0's and 1's.

B_n = the set of all bitstrings of length n.

(To generate a bitstring $b_1 \ldots b_n$).
For $i=1$ to n.
- choose the value (0 or 1) of b_i.

return $b_1 \ldots b_n$.

n-step procedure.

$N_1 = 2 = N_2 = N_3 = \cdots = N_n$.

The number of ways to execute the procedure is

$$N_1 \times N_2 \times N_3 \cdots N_n = \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n} = 2^n.$$

$$\cancel{b_1 \ldots b_n}\ P_1 = b_1 b_2 \cdots \overset{1}{b_i} \cdots b_n.$$

$$P_2 = b_1 b_2 \cdots \underset{0}{b_i'} \qquad \Rightarrow$$

$|B_n| = 2^n$.

$f: A \to B$ is one-to-one (injective) if for any distinct $x, y \in A$

$f(x) \neq f(y)$.

How many one-to-one functions are there from a set A of size m onto a set B of size n.

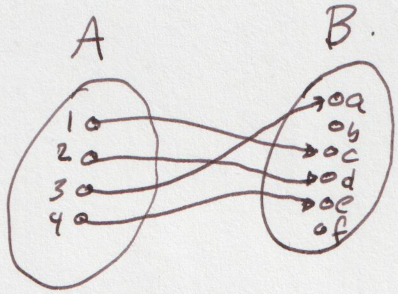
Answer (partial): If $|A| > |B|$ there are ~~zero~~ $\emptyset$ one-to-one functions $f: A \to B$.

Let $x_1 \dots x_m$ be the elements of A.

For $i = 1$ to m.

- choose $f(x_i)$ from the set
  $A \setminus \{f(x_1), f(x_2), \dots, f(x_{i-1})\}$.

return f.



$f(1) = c$
$f(2) = d$
$f(3) = a$
$f(4) = e$

$N_1 = n$.

$N_2 = n - 1$.

$\vdots$

$N_i = n - (i-1) = n - i + 1$.

$\vdots$

$N_m = n - (m-1) = n - m + 1$.

$N_1 = 6 = n$

$N_2 = 5 = n - 1$

$N_3 = 4 = n - 2$

$N_4 = 3 = n - 3$.

The number of ways to execute this procedure is and the number of one-to-one functions $f: A \to B$ is also

$$N_1 \cdot N_2 \cdots N_m = \frac{n(n-1)(n-2) \cdots (n-m+1)(n-m)(n-m-1) \cdots 2 \cdot 1}{(n-m)(n-m-1) \cdots 2 \cdot 1}$$

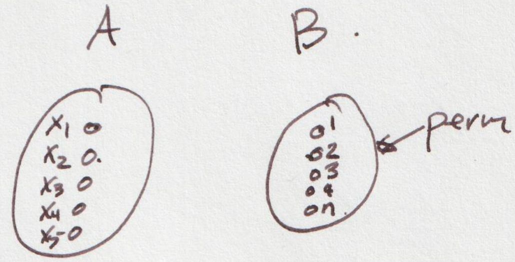
$$= \frac{n!}{(n-m)!}$$

n

$$n! = \begin{cases} n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot 2\cdot 1 & \text{if } n>0\ [n\geq 1]. \\ 1 & \text{if } n=0. \\ \text{undefined} & \text{if } n<0. \end{cases}$$

"n factorial"

$n=m.$



$$\frac{n!}{(n-m)!} = \frac{n!}{0!} = n!$$