Carleton University

Final
Examination
Fall 2024

DURATION: 2 HOURS

No. of students: 500+

Department Name & Course Number: **Computer Science COMP 2804A/B**
Course Instructor: Pat Morin

Authorized memoranda:
Calculator

Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to the proctor. This question paper has **??** pages (not including the cover page).
This examination question paper MAY be taken from the examination room.

In addition to this question paper, students require:

an examination booklet:	no
a Scantron sheet:	yes

Instructions:

1. All questions must be answered on the scantron sheet.
2. Write your name and student number on the scantron sheet.
3. You do not have to hand in this examination paper.
4. Calculators are allowed.

Marking scheme: Each of the 25 questions is worth 1 mark.

Some reminders:

- Young Gauss: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
- Binomial Coefficients: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Newton: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$
- Geometric Series: For $0 < x < 1$, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$
- Geometric distribution: Assume an experiment has a success probability of p. We perform the experiment until it is successful for the first time. The expected number of times we perform the experiment is $1/p$.
- Expected value: $\mathbf{E}(X) = \sum_k k \cdot \Pr(X = k)$. The sum ranges over all values k that X can take.

Question 1. Let X be a set of size n. How many triples (A, B, C) are there such that $A \subseteq X$, $B \subseteq X$, $C \subseteq X$ and A, B, and C are pairwise disjoint? (Pairwise disjoint means that $A \cap B = \emptyset$, $B \cap C = \emptyset$ and $A \cap C = \emptyset$).

(a) 2^n

(b) 3^n

(c) 4^n

(d) $(2^n)^3$

(e) $(\sum_{k=0}^{n} \binom{n}{k})^3$

Question 2. Consider strings of length 85, in which each character is one of the letters a, b, c, d. How many such strings contain exactly 10 occurrences of the letter a?

(a) $\binom{85}{10} \cdot 3^{75}$

(b) $\binom{85}{10} \cdot 3^{85}$

(c) $\binom{85}{10} \cdot 4^{75}$

(d) $\binom{85}{10} \cdot 4^{85}$

(e) None of the above

Question 3. Consider strings of length 85, in which each character is one of the letters a, b, c, d. How many such strings have exactly 15 occurrences of the letter a and exactly 30 occurrences of the letter d?

(a) $\binom{85}{30} \cdot \binom{55}{15} \cdot 2^{40}$

(b) $\binom{85}{30} \cdot \binom{55}{15} \cdot 3^{40}$

(c) $\binom{85}{30} \cdot \binom{55}{15} \cdot 4^{40}$

(d) $\binom{85}{30} \cdot \binom{55}{15} \cdot 5^{40}$

(e) None of the above

Question 4. Consider strings of length 85, in which each character is one of the letters a, b, c, d. How many such strings have exactly 15 letters a or exactly 30 letters d?

(a) $\binom{85}{15} \cdot 3^{70} + \binom{85}{30} \cdot 3^{55}$

(b) $\binom{85}{15} \cdot 3^{70} + \binom{85}{30} \cdot 3^{55} - \binom{85}{15} \cdot \binom{70}{30} \cdot 2^{40}$

(c) $\binom{85}{15} \cdot 4^{70} + \binom{85}{30} \cdot 4^{55}$

(d) None of the above

Question 5. Professor M's class has 20 International students and 50 Canadian students. How many subsets of these students are there that contain exactly 12 International students (and any number of Canadian students) or exactly 12 Canadian students (and any number of Internationl students)?

(a) $\binom{20}{12} + \binom{50}{12}$

(b) $\binom{20}{12} + \binom{50}{12} - \binom{20}{12} \cdot \binom{50}{12}$

(c) $\binom{20}{12} \cdot 2^{50} + \binom{50}{12} \cdot 2^{20}$

(d) $\binom{20}{12} \cdot 2^{50} + \binom{50}{12} \cdot 2^{20} - \binom{20}{12} \cdot \binom{50}{12}$

Question 6. Professor M's class has 20 International students and 50 Canadian students. How many subsets of these students are there that contain at least 3 International students (and any number of Canadian students)?

(a) $2^{70} - 2^{50} - 20 - \binom{20}{2}$

(b) $2^{70} - (21 + \binom{20}{2})2^{50}$

(c) $2^{70} - 2^{50} - 20 \cdot 2^{50}$

(d) None of the above

Question 7. Consider the following statement: For any integers $m \geq 2$ and $n \geq 2$,

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + m \cdot n \ .$$

(a) The statement is false

(b) The statement is true

(c) None of the above

Question 8. Christmas is coming and I'm buying 25 scarves to give away. The scarves come in 7 different colours. How many ways are there to choose the colours of the scarves I will buy?

(a) $\binom{31}{6}$

(b) $\binom{31}{7}$

(c) $\binom{32}{6}$

(d) $\binom{32}{7}$

Question 9. A string that is obtained by rearranging the letters of the word CHEESY is called *wicked*, if the string does not contain the substring EE. Thus, CHEYSE is wicked, whereas SEECHY is not wicked. What is the number of wicked strings?
(a) $6 \cdot \binom{5}{2} \cdot 3 \cdot 2$

(b) $\left(6 \cdot \binom{5}{2} \cdot 3 \cdot 2\right) - 5!$

(c) $6! - 5!$

(d) $\left(6 \cdot \binom{5}{2} \cdot 3\right) - 5!$

Question 10. Consider bitstrings that do not contain 110. Let S_n be the number of such strings having length n. Which of the following is true for any $n \geq 4$?
(a) $S_n = S_{n-1} + S_{n-2} + S_{n-3}$

(b) $S_n = S_{n-1} + S_{n-2}$

(c) $S_n = S_{n-1} + S_{n-2} + 1$

(d) S_n is the nth Fibonacci number f_n

(e) S_n is the $(n+1)$th Fibonacci number f_{n+1}

Question 11. Consider the recursive algorithm IFeelLikeSinging, which takes as input an integer $n \geq 0$:

```
IFeelLikeSinging(n):
  if n = 0 or n = 1 then
    sing O Canada
  else if n is odd then
    IFeelLikeSinging(n + 1)
  else
    IFeelLikeSinging(n/2)
    IFeelLikeSinging(n/2)
```

For any non-negative integer n, let $P(n)$ be the number of times you sing O Canada when you run algorithm IFeelLikeSinging(n). Which of the following statements is true about $P(n)$, for any $n \geq 3$?
(a) $P(n) = n$

(b) $P(n) = 2 \cdot P(n/2)$

(c) $P(n) = 2 \cdot P(\lceil n/2 \rceil)$

(d) $P(n) = 2 \cdot P(\lfloor n/2 \rfloor)$

(e) None of the above

Question 12. You write a multiple-choice exam that has 25 questions. For each question, five options are given to you, and exactly one of these options is the correct answer. Assume that you answer each question by choosing one of the five options uniformly at random, where the choices for different questions are independent of each other. What is the probability that you have exactly 17 correct answers?

(a) $\binom{25}{17} \cdot (1/5)^{17}$

(b) $\binom{25}{27} \cdot (4/5)^{17}$

(c) $\binom{25}{17} \cdot (1/5)^{17} \cdot (4/5)^{8}$

(d) $\binom{25}{17} \cdot (1/5)^{8} \cdot (4/5)^{17}$

Question 13. Consider a uniformly random bitstring of length 5. Define the events

$$A = \text{“the first three bits are 111 or 110”} \ ,$$

$$B = \text{“the last three bits are 111”} \ .$$

Which of the following is true?

(a) The events A and B are independent.

(b) The events A and B are not independent.

(c) None of the above.

Question 14. 500 students are taking this exam. 50 of these students are hiding unauthorized memoranda. Before the exam begins, I choose 100 students at random and thoroughly search each one for unauthorized memoranda. What is the expected number of students that I will find with unauthorized memoranda?

(a) 2

(b) 5

(c) 10

(d) 20

Question 15. Continuing from the previous question, what is the probability that I will not find any student with unauthorized memoranda?

(a) 0

(b) $\binom{400}{100} / \binom{500}{100}$

(c) $\binom{450}{100} / \binom{500}{100}$

(d) $\binom{450}{50} / \binom{500}{100}$

(e) $\binom{400}{50} / \binom{500}{100}$

Question 16. A bowl contains n red balls and m blue balls. We choose 2 balls uniformly at random. Define the events

$$A = \text{"both balls are red"},$$

$$B = \text{"both balls have the same color"}.$$

What is the conditional probability $\Pr(A \mid B)$?

(a) $\frac{\binom{n}{2}}{\binom{n+m}{2}}$

(b) $\frac{\binom{n}{2}}{\binom{n}{2}+\binom{m}{2}}$

(c) $\frac{\binom{n}{2}+\binom{m}{2}}{\binom{m}{2}}$

(d) $\frac{\binom{n}{2}}{\binom{n}{2}\cdot\binom{m}{2}}$

Question 17. We choose an element x uniformly at random from the set $\{1, \ldots, n\}$ for some positive integer n. Consider the events

$$A = \text{"}x \text{ is divisible by 7"},$$

$$B = \text{"}x \text{ is divisible by 3"}.$$

Which of the following is true:

(a) A and B are always independent.

(b) A and B are independent if and only if n is divisible by 21.

(c) A and B are independent if and only if n is disible by 21 or $n < 7$.

(d) A and B are independent if and only if $n \bmod 21 \in \{0, 3, 6\}$.

(e) None of the above.

Question 18. We choose an element x uniformly at random from the set $\{1, 2, 3, \ldots, n\}$ for some positive integer n. Define the events

$$A = \text{"}x \text{ is even"},$$

$$B = \text{"}1 \leq x \leq n/2\text{"}.$$

Which of the following is true?

(a) The events A and B are independent if n is even.

(b) The events A and B are independent if n is odd.

(c) The events A and B are not independent, for any choice of n.

(d) None of the above.

Question 19. You toss a fair coin once and get an outcome $\omega \in \{H, T\}$. Then you give the coin to me and I toss it repeatedly until the first time I get the same outcome, ω, that you got. Define the events

$$A = \text{"my first coin toss is } H\text{"},$$

$$B = \text{"I toss the coin at least twice and my second coin toss is } H\text{"}.$$

Which of the following is true?
(a) The events A and B are independent.

(b) The events A and B are not independent.

(c) None of the above.

Question 20. I flip two fair and independent coins. If the first coin comes up tails, you lose \$1 (i.e., you win $-\$1$). If the second coin comes up heads, you win \$3. (Thus, if the first coin comes up tails and the second coins comes up heads, you win \$2.) Define the random variable X to be the amount of dollars that you win. What is the expected value of X?
(a) 2

(b) 1

(c) 1/4

(d) 1/2

Question 21. Let $n \geq 2$ be an integer. Consider a string $c_1, c_2, \ldots, c_n$ of length n, in which each character c_i is a uniformly random element of the set $\{1, 2, 3\}$ (independently of all other characters). Consider the random variable X whose value is the number of indices $i \in \{1, \ldots, n-1\}$ for which the product $c_i \cdot c_{i+1}$ is even.
What is the expected value $\mathbf{E}(X)$ of the random variable X?
Hint: Use indicator random variables.

1. $\frac{4}{9} \cdot n$
2. $\frac{4}{9} \cdot (n-1)$
3. $\frac{1}{9} \cdot n$
4. $\frac{1}{9} \cdot (n-1)$

Question 22. You are given a fair 20-sided red die and a fair 20-sided blue die. Consider the following experiment:
Experiment: Roll each die once and take the sum of the two rolls. You repeat this experiment until the sum of the two rolls is equal to 21. Consider the random variable

$$X = \text{the number of times you do the experiment.}$$

(This value X includes the experiment in which the sum is 21 for the first time.) What is the expected value $\mathbf{E}(X)$ of the random variable X?
(a) 40

(b) 30

(c) 20

(d) 10

(e) 5

Question 23. Is the following statement true or false?

$$\text{For any random variable } X,\ \mathbf{E}(X^2) = (\mathbf{E}(X))^2.$$

(a) The statement is true.

(b) The statement is false.

(c) None of the above.

Question 24. A gumball machine starts out with 700 red gumballs and 300 blue gumballs for a total of 1000 gumballs. When you put a quarter in the machine a random gumball comes out, leaving 999 gumballs in the machine. If you put in another quarter, then you get a another random gumball from these 999 gumballs, leaving 998 gumballs in the machine. And so on.
Suppose you arrive at the gumball machine and put in ten quarters, so that you get 10 gumballs. What is the expected number of red gumballs you get?
(a) $(700 \cdot 699 \cdot \cdots \cdot 691)/(1000 \cdot 999 \cdot \cdots \cdot 991)$

(b) 6.99

(c) $700^{10}/(1000 \cdot 999 \cdot \cdots \cdot 991)$

(d) 7

(e) None of the other answers is correct

Question 25. How do you feel about writing an exam at 9:00am on a Friday morning?

(a) I would rather be sleeping in and then having brunch.

(b) I would rather be sleeping in and then having brunch.

(c) I would rather be sleeping in and then having brunch.

(d) I would rather be sleeping in and then having brunch.

(e) I would rather be sleeping in and then having brunch.

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