Important: This is your personal exam. Do not share it with anyone. If a copy of any portion of this exam is ever found online it will be considered an academic violation and will be dealt with as such by the Dean’s Office.

This is a closed book exam. You are to do this exam on your own without consulting anyone else or using the internet.

Submit your answers at this URL: https://forms.gle/1mDZw4etNvKst8CK9
(Copy and paste the link into your browser if necessary.)

Marking Scheme: Each of the 17 questions is worth 1 mark.

Reminders:

• Binomial coefficients:

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

• Newton’s Binomial Theorem:

\[ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \]

• Fibonacci numbers:

\[ f_n = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
f_{n-1} + f_{n-2} & \text{if } n \geq 2 
\end{cases} \]
Question 1 (b). The School of Computer Science has $f$ full professors, $a$ associate professors, $t$ assistant professors, $i$ instructors, and $s$ students. The SCS Executive Committee must consist of

- 2 full ($f$) professors;
- 1 or 2 associate ($a$) professor;
- 1 assistant ($t$) professor;
- 1 instructor ($i$); and
- 2 students ($s$).

How many ways are there to form an SCS Executive Committee?

(1) $\binom{f}{2} \cdot \binom{a}{2} \cdot t \cdot i \cdot \binom{s}{2}$
(2) $\binom{f}{2} \cdot a \cdot t \cdot i \cdot \binom{s}{2}$
(3) $\binom{f}{2} \cdot a^2 \cdot t \cdot i \cdot \binom{s}{2}$
(4) $\binom{f}{2} \cdot 2a \cdot t \cdot i \cdot \binom{s}{2}$
(5) $\binom{f}{2} \cdot a \cdot t \cdot i \cdot \binom{s}{2} + \binom{f}{2} \cdot \binom{a}{2} \cdot t \cdot i \cdot \binom{s}{2}$

Question 2 (b). A class contains $n \geq 2$ distinct students and wants to send a group of $k \leq n - 2$ students on a field trip. Two of these students, Fred and Steve, are bullies. The rest of the students are not bullies.

How many ways are there to choose a set of $k$ students that does not contain two bullies?

(1) $\binom{n-1}{k} + \binom{n-2}{k}$
(2) $\binom{n-2}{k}$
(3) $\binom{n-2}{k-1} \cdot (n-k-1)$
(4) $\binom{n-1}{k}$
(5) $\binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1}$

Question 3 (b). Piper ($P$) and Marley ($M$) are getting married. In addition to Piper and Marley, the wedding party has $p$ of Piper’s friends $P_1, \ldots, P_p$ and $m$ of Marley’s friends $M_1, \ldots, M_m$. Piper and Marley have no common friends, so $\{P_1, \ldots, P_p\} \cap \{M_1, \ldots, M_m\} = \emptyset$.

It’s later in the evening, everyone has been drinking, and Piper and Marley had a fight. It’s time to take a wedding photo and we want to line up the entire wedding party without having Piper and Marley standing next to each other. For example we could line them up like $M, P_1, \ldots, P_p, P, M_1, \ldots, M_m$.

How many ways are there to line up to the group so that Piper and Marley are not standing beside each other?

(1) $p!m!$
(2) $2p!m!$
(3) $4p!m!$
(4) $p!m! + p!m!$
(5) $(p + m + 1)!(p + m)$
Question 4 (b). Let’s go to the animal shelter to take home some cats. At the shelter there are $b \geq 5$ black cats $B_1, \ldots, B_b$ and $w \geq 5$ white cats $W_1, \ldots, W_w$. All the cats are distinct; we can distinguish between any two cats, even if they have the same colour.

How many ways are there to take home 5 cats so that we take home an odd number of white cats?

$$
(1) \begin{aligned}
&\binom{b+w}{5} \\
&(2) \binom{b}{5} + \binom{b}{3} \cdot \binom{w}{2} + \binom{w}{1} \cdot \binom{b}{4} \\
&(3) \binom{w}{5} + \binom{w}{3} \cdot \binom{b}{2} + \binom{b}{1} \cdot \binom{w}{4} \\
&(4) \binom{w}{5} + \binom{w}{4} \cdot b + \binom{w}{3} \cdot \binom{b}{2} \\
&(5) \sum_{k=0}^{5} \binom{w}{k} \cdot \binom{b}{5-k}
\end{aligned}
$$

Question 5 (b). Let $n \geq 5$. How many strings of length $n$ over the alphabet $\{a, b, c\}$ begin with $abc$ or end with $bb$?

$$
(1) 3^n \\
(2) 3^{n-5} \\
(3) 3^{n-2} - 3^{n-5} \\
(4) 3^{n-3} + 3^{n-2} - 3^{n-5} \\
(5) 3^n - 3^{n/2}
$$

Question 6 (b). How many strings can be obtained by rearranging the letters of the word SIMSANTEETERS

$$
(1) 13! \\
(2) 13!/48 \\
(3) 13 \cdot 12 \cdot \binom{11}{3} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1 \\
(4) 13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{3} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1 \\
(5) 13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{2} \cdot \binom{7}{3} \cdot \binom{5}{3} \cdot 2 \cdot 1
$$

Question 7 (b). What does $\sum_{k=1}^{n} \binom{n}{k} 2^{n-k}$ count?

$$
(1) \text{The number of strings of length } n \text{ over the alphabet } \{a, b\} \\
(2) \text{The number of strings of length } n \text{ over the alphabet } \{a, b\} \text{ that contain at least one } a \\
(3) \text{The number of strings of length } n \text{ over the alphabet } \{a, b, c\} \\
(4) \text{The number of strings of length } n \text{ over the alphabet } \{a, b, c\} \text{ that contain no } a \\
(5) \text{The number of strings of length } n \text{ over the alphabet } \{a, b, c\} \text{ that contain at least one } a
$$
Question 8 (b). I have a jar with 55 balls numbered 1, . . . , 55. I want to take balls out of the jar until I find two different pairs of balls \( \{b_1, b_2\} \) and \( \{b_3, b_4\} \) such that \( b_1 - b_2 = b_3 - b_4 \).

The fewest balls I must take out before I am guaranteed this will happen is:

(1) 4 balls
(2) 11 balls
(3) 15 balls
(4) 27 balls
(5) 55 balls

Question 9 (b). What is the coefficient of \( x^{13}y^9 \) in the expansion of \( (3x - 2y)^{22} \)?

(1) \( \binom{22}{1} \cdot 3^{13} \cdot 2^9 \)
(2) \( -\binom{22}{1} \cdot 3^{13} \cdot 2^9 \)
(3) \( \binom{22}{12} \cdot 3^{13} \cdot 2^9 \)
(4) \( -\binom{22}{12} \cdot 3^{13} \cdot 2^9 \)
(5) None of the other answers is correct

Question 10 (b). A string over the alphabet \( \{a, b, c\} \) is called great if it does not contain \( bc \) or \( ba \). Let \( n \geq 2 \). How many great strings of length \( n \) are there?

(1) \( 2^n \)
(2) \( 2^{n+1} - 1 \)
(3) \( 3^n \)
(4) \( 3^n - 2^n \)
(5) \( f_{n+1} \)

Question 11 (b). A bitstring is called 00-free if it does not contain two 0s next to each other. In class we have seen that, for any \( m \geq 1 \), the number of 00-free bitstrings of length \( m \) is equal to the \( (m + 2) \)th Fibonacci number \( f_{m+2} \).

What is the number of 00-free bitstrings of length 30 that have 1 at position 9? (The positions are numbered 1, 2, . . . , 30.)

(1) \( f_7 \cdot f_{20} \)
(2) \( f_8 \cdot f_{21} \)
(3) \( f_9 \cdot f_{22} \)
(4) \( f_{10} \cdot f_{23} \)
(5) None of the other answers is correct
Question 12 (b). A string over the alphabet \{x, y, z\} is called fabulous if it does not contain \(xyz\), \(xyx\), or \(xx\). For \(n \geq 1\), let \(A_n\) denote the number of fabulous strings of length \(n\). Which of the following is true for any \(n \geq 4\)?

(1) \(A_n = A_{n-1} + A_{n-2} + A_{n-3}\)
(2) \(A_n = 2A_{n-1} + A_{n-2} + A_{n-3}\)
(3) \(A_n = 2A_{n-1} + 2A_{n-2} + A_{n-3}\)
(4) \(A_n = 2A_{n-1} + 2A_{n-2} + 2A_{n-3}\)
(5) None of the other answers is correct

Question 13 (b). Consider the following recursive function \(\text{BAR}(n)\):

\[
\text{BAR}(n) :
\begin{align*}
\text{if } n \leq 3 & \text{ then} \\
\text{return } n \\
\text{return } \text{BAR}(n - 1) + \text{BAR}(n - 3)
\end{align*}
\]

When running \(\text{BAR}(46)\) how many calls are there to \(\text{FU}(41)\)?

(1) 4
(2) 6
(3) 8
(4) 9
(5) 10

Question 14 (b). You are given an infinite supply of red marbles and an infinite supply of blue marbles. Let \(S_n\) be the number of ways of placing \(n\) of these marbles in a line so that you never have three red marbles in a row. Which of the following is true, for any \(n \geq 3\)?

(1) \(S_n = 2S_{n-1}\)
(2) \(S_n = 4S_{n-3}\)
(3) \(S_n = S_{n-1} + S_{n-2} + S_{n-3}\)
(4) \(S_n = S_{n-1} + 2S_{n-2} + S_{n-3}\)
(5) \(S_n = 2^n\)

Question 15 (b). You toss a fair coin 12 times. Define the event:

\[A = \text{“the results of the last three flips are equal”}\]

What is \(\text{Pr}(A)\)?

(1) \(1/2\)
(2) \(1/4\)
(3) \(1/6\)
(4) \(1/8\)
(5) \(1/16\)
Question 16 (b). A bag contains $r$ red balls, $b$ blue balls, and $g$ green balls. We reach into the bag and choose a uniformly random subset of 2 balls. Define the event:

$$A = \text{“this subset has no blue balls”}$$

What is $\Pr(A)$?

1. $1/(\binom{r+g+b}{2})$
2. $rg/(\binom{r+g+b}{2})$
3. $(\binom{r+g}{2})/(\binom{r+g+b}{2})$
4. $(r + g)^2/(r + g + b)^2$
5. $rg/rgb$

Question 17 (b). A bag contains $r$ red balls, $b$ blue balls, and $g$ green balls. We reach into the bag and choose a uniformly random subset of 2 balls. Define the event:

$$A = \text{“the subset contains one red ball and one green ball”}$$

What is $\Pr(A)$?

1. $1/(\binom{r+g+b}{2})$
2. $rg/(\binom{r+g+b}{2})$
3. $(\binom{r+g}{2})/(\binom{r+g+b}{2})$
4. $(r + g)^2/(r + g + b)^2$
5. $rg/rgb$