**COMP3804 Winter 2006 — Assignment 1 — Due Thursday Jan. 19**

This assignment is due in class on Thursday January 19th. If you are not able to come to class then please put your assignment in the COMP3804 assignment box in the CCSS lounge. The assignments will be picked up from this box immediately after class.

# 1 Asymptotic Notation

In the following questions, all the functions ($f$, $g$, and $h$) we consider are increasing non-negative functions of $n$. All logarithms are base 2 unless stated otherwise.

## 1.1 Order of Growth.

List the following functions in increasing order of asymptotic complexity. Do this by using the notation $f(n) \ll g(n)$ to denote that the function $f(n) = o(g(n))$ and using the notation $f(n) \equiv g(n)$ to denote that $f(n) = \Theta(g(n))$. All logarithms are base 2. For example,

$$\log\log n \ll \log n \equiv 2\log n \ll n^{1/2} \ll n \equiv 10^5 n \ll n^2 \ .$$

Just listing the functions as above will suffice. There is no need to show your work. (Hint: You may need to lookup Stirling's inequality in your textbook.)

| | | | | |
|---|---|---|---|---|
| $(\log n)^3$ | $4^{\log n}$ | $\log n$ | $\log^* n$ | $n!$ |
| $e^n$ | $2^n$ | $\log\log n$ | $(\log n)/(\log\log n)$ | $2^{\log n}$ |
| $n$ | $10^5 \log n$ | $n^2$ | $n^n$ | |

## 1.2 Application of the Definition

Consider the functions $f(n) = \log n$, $g(n) = 2^{\sqrt{\log n}}$ and $h(n) = n^{0.001}$. Prove, using the definition of $o$, that $f(n) \ll g(n) \ll h(n)$. Show all your work.

## 1.3 Relationship with Limits

In Calculus, we use the notation $\lim_{n\to\infty} f(n) = c$ if, for every value $\epsilon > 0$, there exists a value $n'$ such that $|f(n) - c| < \epsilon$ for all $n \geq n'$. In words, $f(n)$ gets closer and closer to $c$ as $n$ gets large. Using the appropriate definitions, prove the following:

1. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some constant $c$ then $f(n) = \Theta(g(n))$.
2. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ then $f(n) = o(g(n))$.

## 2 Summations

### 2.1 Exact Closed-Form Solutions

Give exact closed-form formulas for the following summations (after manipulating the sum a bit, the solutions can be found in the chapter/appendix on summations in your textbook). You may use the notation $H_n$ as a "closed form" for $\sum_{i=1}^{n} 1/i$. Show your work.

As an example, to evaluate the sum $\sum_{i=1}^{n} 2i$ we could do

$$\begin{aligned}\sum_{i=1}^{n} 2i &= 2\sum_{i=1}^{n} i &(1)\\ &= 2n(n+1)/2 &(2)\\ &= n(n+1)\ . &(3)\\ & &(4)\end{aligned}$$

(Step (2) uses the identity $\sum_{i=1}^{n} i = n(n+1)/2$ that appears on page 1059 of the textbook.)

1. $\sum_{i=1}^{n}(i+1)$
2. $\sum_{i=1}^{n}(2i+1/i)$
3. $\sum_{i=1}^{n} n/i$
4. $\sum_{i=0}^{\infty} 1/2^i$
5. $\sum_{i=1}^{\infty} 1/2^i$
6. $\sum_{i=a}^{b} 1/i$
7. $\sum_{i=0}^{n}(-1)^i/2^i$

### 2.2 Asymptotic Upper Bounds

Give asymptotic (big $O$) upper bounds on the following summations. For example, to bound $\sum_{i=1}^{n} i$ we might do

$$\sum_{i=1}^{n} i \leq \sum_{i=1}^{n} n = n^2 = O(n^2)$$

1. $\sum_{i=1}^{n} 2$
2. $\sum_{i=1}^{n} 10/i$
3. $\sum_{i=1}^{n} i^4$
4. $\sum_{i=1}^{n} n/(3^i)$
5. $\sum_{i=1}^{\log_2 n} 2^i$
6. $\sum_{i=1}^{\log_2 n} 3^i$

### 2.3 Bounding by Integrals (Part I)

Recall that, for $c > 0$, the integral $\int_0^n ax^c\,dx = an^{c+1}/(c+1)$.

1. Consider the sum $\sum_{i=1}^n i^2$. Draw a picture that illustrates the relationship between this sum and the integral $\int_0^n x^2\,dx$.

2. Using what you discovered above, prove that

$$n^3/3 \le \sum_{i=1}^n i^2 \le n^3/3 + n^2 \enspace .$$

3. What can you deduce about the sum $\sum_{i=1}^n i^c$, where $c$ is any constant greater than 1?

### 2.4 Bounding by Integrals (Part II)

For this question you will need your textbook and a calculus book. In the calculus book, lookup the integral $\int 1/x^2\,dx$. In your textbook, find the section on bounding harmonic numbers using integrals. You will find a proof that

$$\ln n \le H_n \le \ln n + 1 \enspace .$$

Use the same technique to prove bounds on the sum $\sum_{i=1}^n 1/i^2$.

## 3 Recurrences

For the following recurrences state (1) the maximum depth of the recursion tree, (2) the total number of nodes at the $i$th level of the recursion tree, (3) the maximum number of leaves in the recursion tree, and (4) the asymptotic solution of the recurrence. For example, the recurrence $T(n) = 3T(n/2) + O(n)$ has

1. a recursion tree of depth at most $\log_2 n$,

2. the total number of nodes at level $i$ is at most $3^i$

3. the maximum number of leaves is at most $3^{\log_2 n}$

4. the recurrence is bounded by the sum

$$\begin{aligned} T(n) &\le \sum_{i=0}^{\log_2 n} 3^i O(n/2^i) && (5) \\ &= O(n) \times \sum_{i=0}^{\log_2 n} (3/2)^i && (6) \\ &= O(n) \times \frac{(3/2)^{\log_2 n+1} - 1}{3/2 - 1} && (7) \end{aligned}$$

$$
\begin{aligned}
&= O(n) \times \frac{(3/2)(3/2)^{\log_2 n} - 1}{3/2 - 1} && (8)\\
&= O(n) \times \frac{(3/2)(3/2)^{\log_2 n} - 1}{1/2} && (9)\\
&= O(n) \times \frac{(3/2)n^{\log_2(3/2)} - 1}{1/2} && (10)\\
&\leq O(n) \times \frac{(3/2)n^{\log_2(3/2)}}{1/2} && (11)\\
&= O(n) \times O(n^{\log_2(3/2)}) = O(n^{1+\log_2(3/2)}) = O(n^{1.585}) && (12)
\end{aligned}
$$

(Note that Step (7) in the above derivation follows from the formula (A.5) for geometric sums given on page 1060 of the textbook. This always happens when evaluating such sums.)

1. $T(n) = 3T(n/3) + O(1)$
2. $T(n) = 3T(n/3) + O(n)$
3. $T(n) = 3T(n/3) + O(n^2)$
4. $T(n) = 5T(n/4) + O(1)$
5. $T(n) = 5T(n/4) + O(n)$
6. $T(n) = T(n/2) + O(1)$
7. $T(n) = T(n/2) + O(n)$
8. $T(n) = 2T(n-1) + O(1)$
9. $T(n) = 2T(n-2) + O(1)$
10. $T(n) = T(n_1) + T(n_2) + O(n)$ where $0 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.
11. $T(n) = T(n_1) + T(n_2) + O(n)$ where $n/3 \leq n_1, n_2 \leq 2n/3$ and $n_1 + n_2 = n$.