Name:________________________________ Student Number:________________________________

COMP4804 ASSIGNMENT 1: DUE WEDNESDAY JANUARY 25, 23:59EDT

Print this assignment and answer all questions in the boxes provided. Any text outside of the boxes will not be considered when marking your assignment.

1 Frequency Assignment in Wireless Networks

We have a graph $G = (V, E)$ in which every vertex has degree 6, $|V| = n$ and $|E| = m$. For each vertex $v \in V$, we color v uniformly (and independently from all other vertices) at random with a color selected from the set $\{1, \dots, k\}$.

1. We say that an edge $e = (u, v)$ is *good* if u and v are assigned different colors in the above experiment and *bad* otherwise. What is the probability that an edge e is bad?

$\frac{1}{k}$

2. What are the expected numbers of bad edges and good edges?

$E(\#\text{ bad edges}) = \frac{m}{k}$ $E(\#\text{ good edges}) = \frac{m(k-1)}{k}$

3. We say that a vertex v is *dead* if all 6 of v's incident edges are bad. What is the probability that a particular vertex v is dead? What is the expected number of dead vertices?

$\frac{1}{k^6}$ $\frac{n}{k^6}$

4. How many colors k do we need if we want the expected number of dead vertices to be at most: (a) $n/10$, (b) $n/100$, and (c) $n/1000$

a) $\frac{n}{k^6} \leq \frac{n}{10}$

$k^6 \geq 10$

$k \geq 2$

b) $k \geq 3$

c) $k \geq 4$

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2 Approximating Max-2-Sat

A 2-CNF formula is the conjunction of a set clauses, where each clause is the disjunction of two (possibly negated, but distinct) variables. For example, the boolean formula

$$(a \vee b) \wedge (b \vee \neg d) \wedge (\neg a \vee c)$$

is a 2-CNF formula with 3 clauses. When we assign truth values to the variables (a, b, c and d above) we say that the assignment *satisfies* the formula if the formula evaluates to true. In general, it is not always possible to satisfy a 2-CNF-Formula, so we may try to satisfy most of the clauses.

1. Describe an analyze a very simple randomized algorithm that takes as input a 2-CNF formula with n clauses and ouputs a truth-assignment such that the expected number of clauses satisfied by the assignment is at least $3n/4$. (Prove that the running time of your algorithm is small and that the expected number of clauses it satisfies is at least $3n/4$. You may assume that the variables are named $a_1, \ldots, a_m$, $m \leq n$, so that you can associate truth values with variables by using an array of length m.)

Let each a_i be an independent variable with

$Pr(a_i = T) = Pr(a_i = F) = \frac{1}{2}$

$Pr(\text{clause} = T) = 3/4$

$\mathbb{E}(\#\text{ of } T \text{ clauses}) = \frac{n \cdot 3}{4}$ via indicators

2. Your algorithm implies something about all 2-CNF formulas having at most 3 clauses. What does it imply?

$\mathbb{E}(\#\text{ of } T \text{ clauses}) = \frac{9}{4} > 2$

Every 2-CNF formula with at most 3 clauses is satisfiable.

3. What does your algorithm guarantee for d-CNF formulas? (Where each clause contains d distinct variables.)

$Pr(d\text{-clause} = T) = 1 - \left(\frac{1}{2}\right)^d$

$\mathbb{E}(\#\text{ of } T \text{ } d\text{-clauses}) = n\left(1 - \frac{1}{2^d}\right)$

Every d-CNF formula with n clauses is satisfiable for $d > \log_2 n$

$\frac{n}{2^d} < 1$; $n < 2^d$; $\log_2 n < d$

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3 Computing the OR of a Bit String

We have are given a bit-string $B_1,\ldots,B_n$ and we want to compute the OR of its bits, i.e., we want to compute $B_1 \vee B_2 \vee \cdots \vee B_n$. Suppose we use the following algorithm to do this:

```
1: for i ← 1,...,n do
2:   if B_i = 1 then
3:     return 1
4: return 0
```

1. In the worst case, what is the number of times line 2 executes, i.e., how many bits must be inspected by the algorithm? Describe an input $B_1,\ldots,B_n$ that achieve the worst case when the output is 0 and when the output is 1.

a) n

b) $0\,0\,0 \ldots 0\,0$

c) $0\,0\,0 \ldots 0\,1$

2. Consider the following modified algorithm:

```
Toss a coin c
if c comes up heads then
  for i ← 1,...,n do
    if B_i = 1 then
      return 1
else
  for i ← n,...,1 do
    if B_i = 1 then
      return 1
return 0
```

Assume that exactly one input bit $B_k = 1$. Then what is the expected number of input bits that the algorithm examines.

$$\mathbb{E}(\#\text{ of bits examined}) = \frac{1}{2}k + \frac{1}{2}(n-k+1) = \frac{n+1}{2}$$

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4 3-Way Partitioning

Suppose you are working on a system where two values can only be compared using the $<$ operator. (Sorting in Python is an example.) Here is an algorithm that, given an array $A[1], \ldots, A[n]$ and a value x classifies the elements of A as either less than, greater than or equal to x.

3-WAY-PARTITION(A, x)

```
1: for i = 1 to n do
2:   if A[i] < x then
3:     add A[i] to S_<
4:   else if A[i] > x then
5:     add A[i] to S_>
6:   else
7:     add A[i] to S_=
```

1. Let $n_<$, $n_>$ and $n_=$ denote the number of elements of A that less than, greater than or equal to x, respectively. State the exact number of comparisons performed by 3-WAY-PARTITION.

$n_< + 2n_> + 2n_=$

2. Show that there exists a randomized algorithm that uses only 1 random bit (coin toss) and performs and expected number of comparisons that is $2n_= + \frac{3}{2}(n_< + n_>)$.

same code:

1:
2:
3:
4:
5:
6:
7:

coin toss

1

0

$$\frac{1}{2}(n_< + 2n_>) + \frac{1}{2}(n_> + 2n_<) + 2n_= =$$

$$= 2n_= + \frac{3}{2}(n_< + n_>)$$

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5

Matchings

1 = lemon, 2 = lime

We have a bag of n candies, $n/2$ of which are lemon and $n/2$ of which are lime. Consider the following experiment: We reach into the bag and pull out two candies. If they're different flavours we eat them both. Otherwise, we put them both back in the bag.

1. What is the probability that we eat the candies? (Warning: It's not exactly 1/2)

$$\Pr(12) = \frac{1}{2}\cdot\frac{n/2}{n-1} = \Pr(21)$$

$$\Pr(12 \text{ or } 21) = \Pr(12)+\Pr(21) = \frac{n}{2(n-1)} > \frac{1}{2}$$

2. What is the expected number of times we have to repeat this experiment until we get to eat some candy?

We have geometric trials until the first success

$$\mathbb{E}(\#\text{ of trials}) = \frac{1}{\Pr(12 \text{ or } 21)} = \frac{2(n-1)}{n} = 2-\frac{2}{n}$$

3. Suppose the number of candies are not the same: There are n_1 limes and n_2 lemons. Then what is the probability that we eat the candies.

$$\Pr(12) = \Pr(1)\cdot\Pr(2|1) = \frac{n_1}{n_1+n_2}\cdot\frac{n_2}{n_1+n_2-1} = \frac{n_2}{n_1+n_2}\cdot\frac{n_1}{n_1+n_2-1} = \Pr(2)\cdot\Pr(1|2) = \Pr(21)$$

$$\Pr(\text{we eat}) = \Pr(12)+\Pr(21) = \frac{2n_1n_2}{(n_1+n_2)(n_1+n_2-1)}$$

4. If we start with a bag containing $n/2$ lime candies and $n/2$ lemon candies, then what is the expected number of times we have to repeat this experiment before the bag is empty? $2-\frac{2}{2}$

$$\frac{2(n-1)}{n}+\frac{2(n-3)}{n-2}+\frac{2(n-5)}{n-4}+\dots+1 = 2-\frac{2}{n}+2-\frac{2}{n-2}+2-\frac{2}{n-4}+\dots+1 =$$

$$= 2\cdot\frac{n}{2} - \left(\frac{1}{n/2}+\frac{1}{\frac{n}{2}-1}+\frac{1}{\frac{n}{2}-2}+\dots+\frac{1}{\frac{n}{2}-\frac{n}{2}+1}\right) \approx n - \log\frac{n}{2}.$$

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5. Show that, if we start with a bag containing $n/3$ lime candies and $2n/3$ lemon candies then the expected number of times we repeat this experiment is $\Omega(n \log n)$. Hint: Harmonic numbers, from Lecture 1 should come up.

$i \leftarrow$ # of limes

$\frac{n}{3}+i \leftarrow$ # of lemons

From 5.3 $\quad P_i = 2 \cdot \frac{i}{\frac{n}{3}+2i} \cdot \frac{\frac{n}{3}+i}{\frac{n}{3}+2i-1}$

$\frac{2}{3} \leq \frac{\frac{n}{3}+i}{\frac{n}{3}+2i-1} \leq 1$

$$P_i \leq \frac{2i}{\frac{n}{3}+2i}$$

$$\mathbb{E}\left(\text{\# of trials before we eat candy, when we have } i \text{ limes}\right) = \frac{1}{P_i} \geq \frac{\frac{n}{3}+2i}{2i} = \frac{n}{6i}+1$$

$$\mathbb{E}\left(\text{\# of trials before we eat all the limes}\right) = \sum_{i=1}^{n/3} \frac{1}{P_i} \geq \sum_{i=1}^{n/3}\left(\frac{n}{6i}+1\right) =$$

$$= \frac{n}{6} \cdot H_{n/3} + \frac{n}{3} \geq$$

$$\geq \frac{n}{6} \ln\left(\frac{n}{3}\right) + \frac{n}{3} = \Omega(n \log n).$$