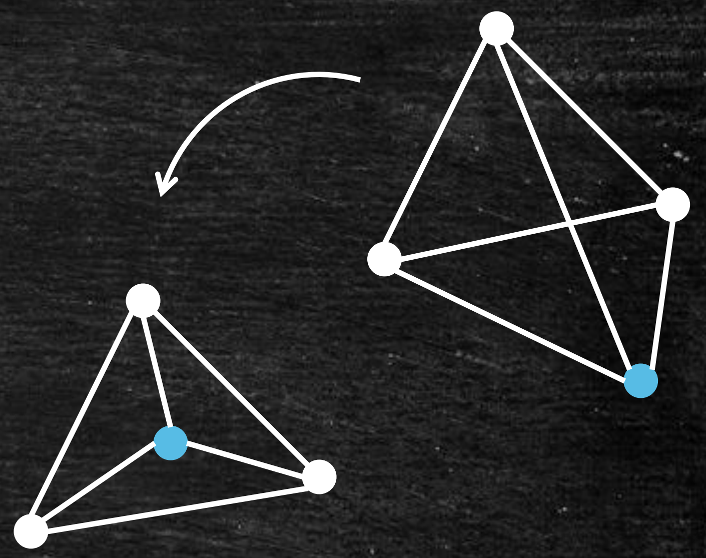


Graph Planarity

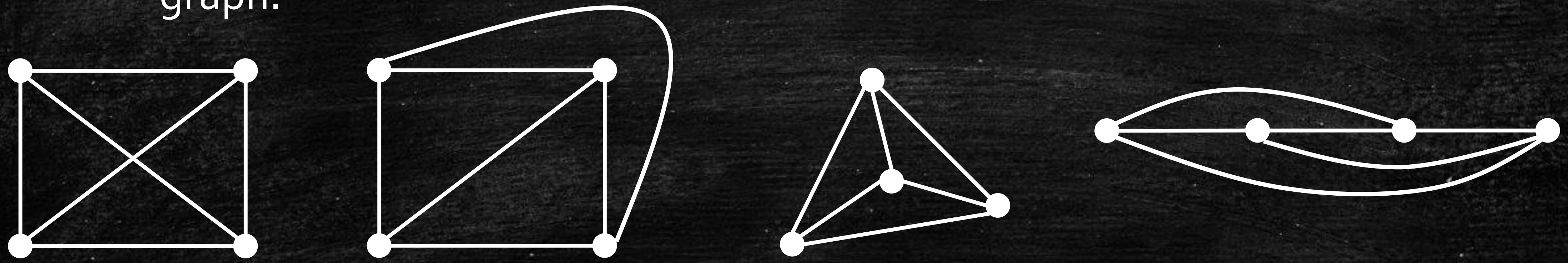
Alina Shaikhet

Outline

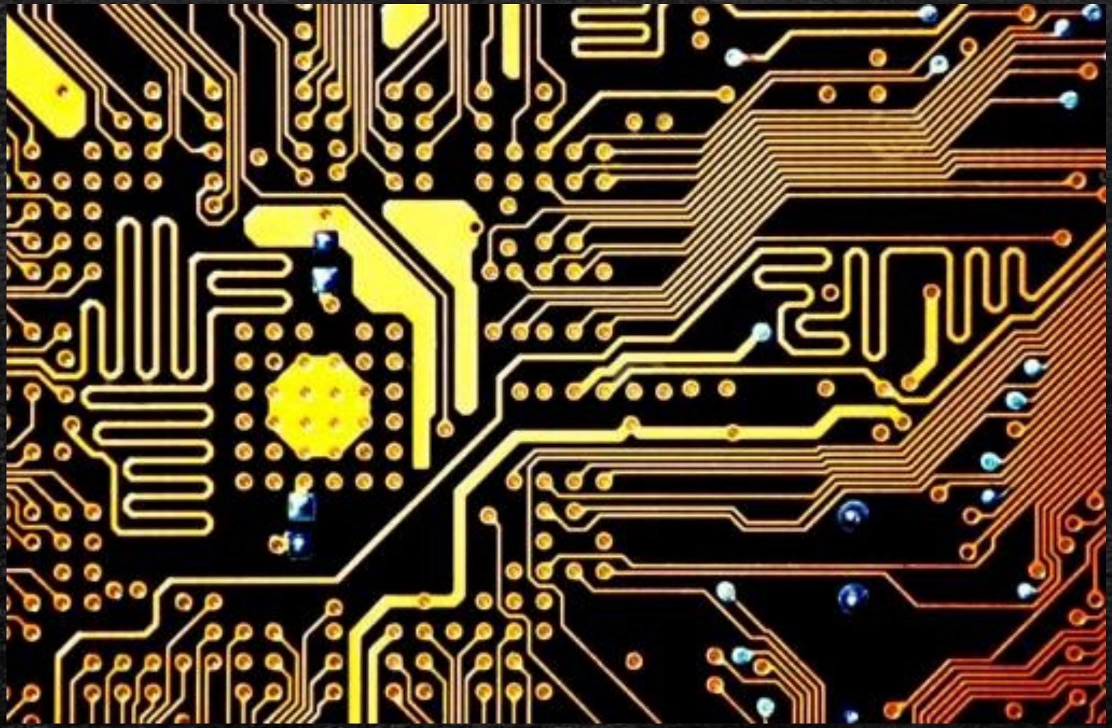
- Definition.
- Motivation.
- Euler’s formula.
- Kuratowski’s theorems.
- Wagner’s theorem.
- Planarity algorithms.
- Properties.
- Crossing Number

Definitions

- A graph is called planar if it can be drawn in a plane without any two edges intersecting.
- Such a drawing we call a planar embedding of the graph.
- A plane graph is a particular planar embedding of a planar graph.

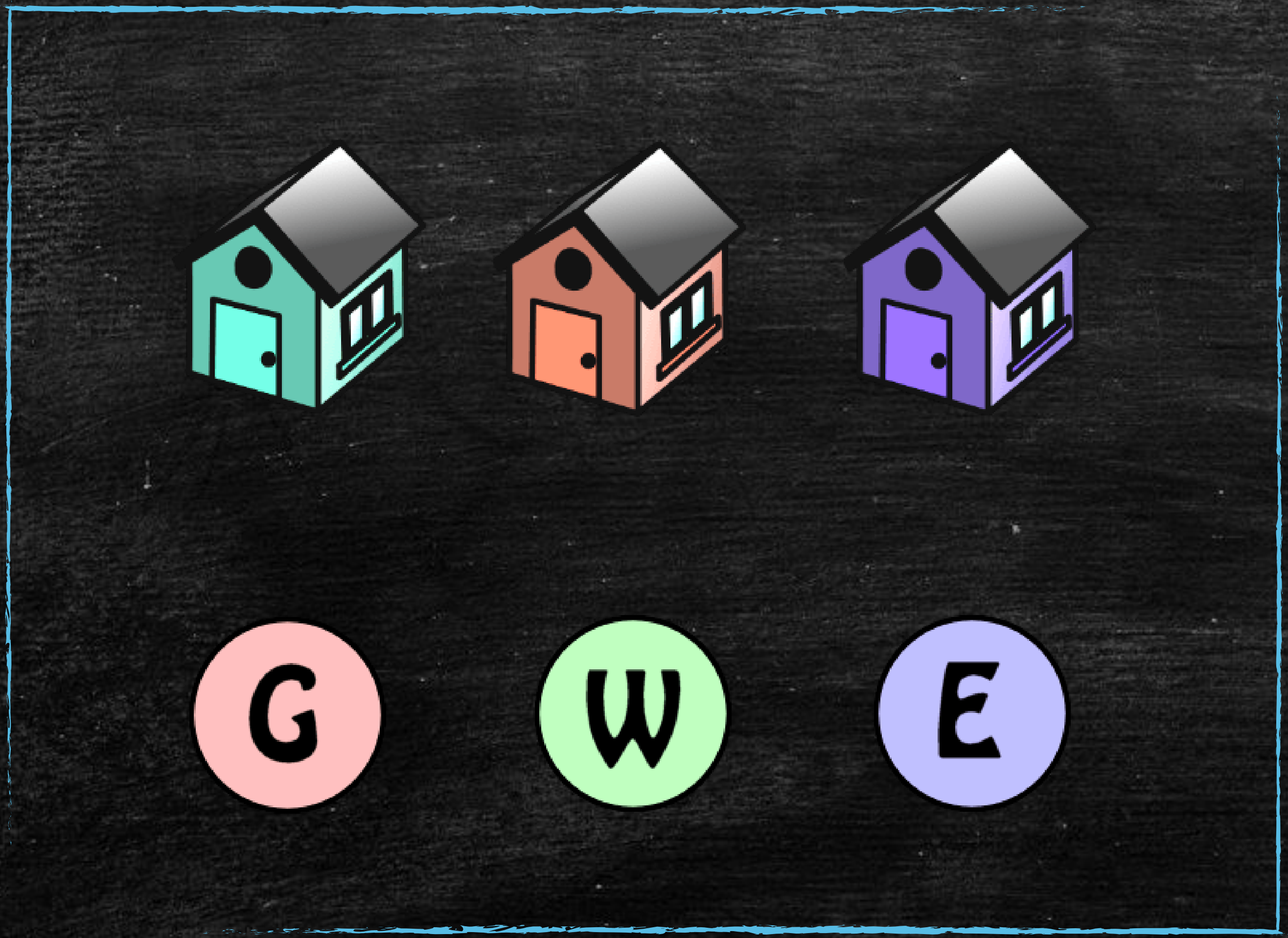


Motivation



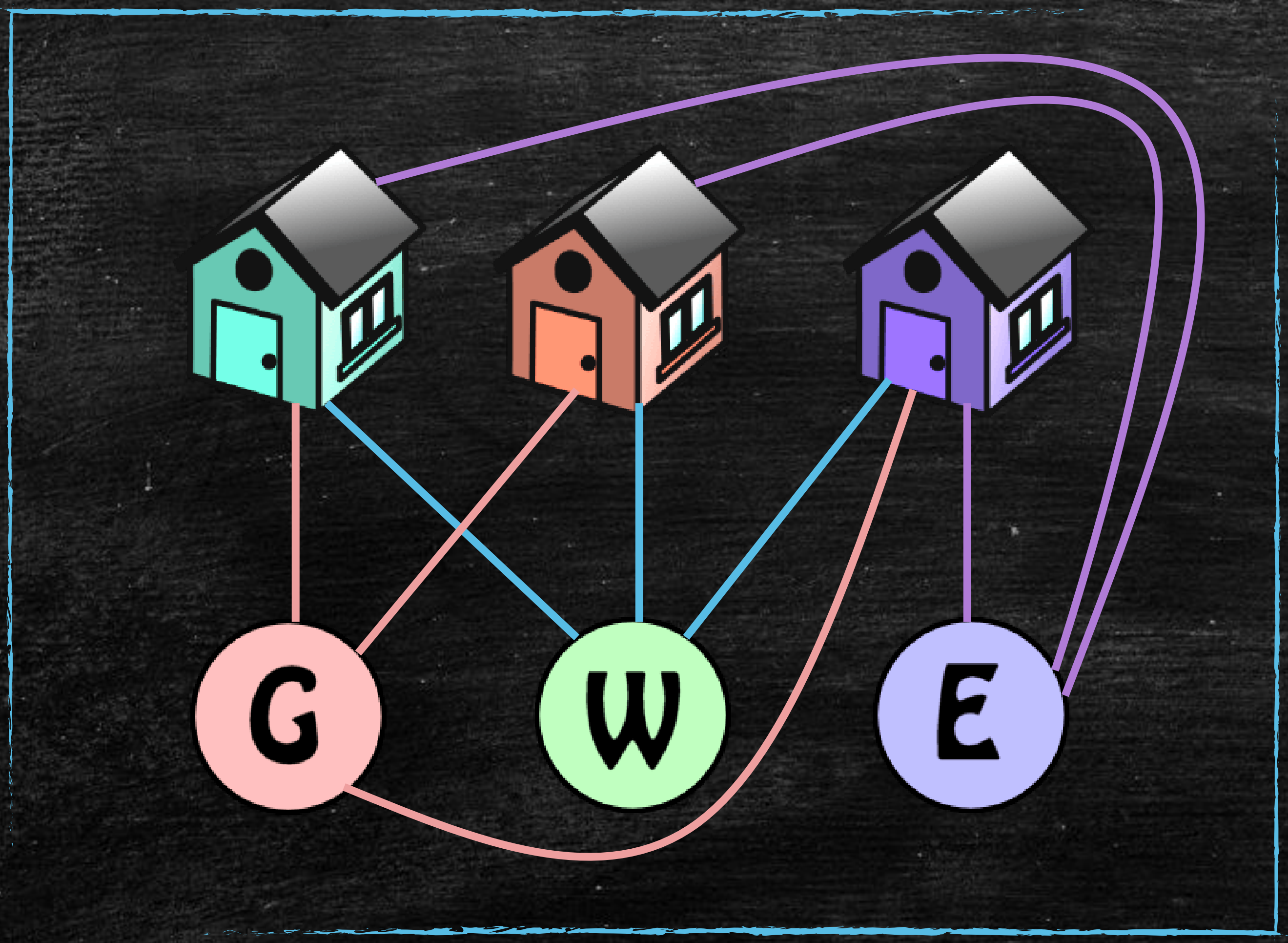
- Circuit boards.

Motivation



- Circuit boards.
- Connecting utilities (electricity, water, gas) to houses.

Motivation



- Circuit boards.
- Connecting utilities (electricity, water, gas) to houses.

Motivation

- Circuit boards.
- Connecting utilities (electricity, water, gas) to houses.
- Highway / Railroads / Subway design.

Euler’s formula.

self loops
multi-edges

Consider any plane embedding of a planar connected graph.

Let V - be the number of vertices,

E - be the number of edges and

F - be the number of faces (including the single unbounded face),

Then $V - E + F = 2$.

Euler formula gives the necessary condition for a graph to be planar.

Euler's formula.

~~self loops~~
~~multi-edges~~

Consider any plane embedding of a planar ~~connected~~ graph.

Let V - be the number of vertices,

E - be the number of edges and

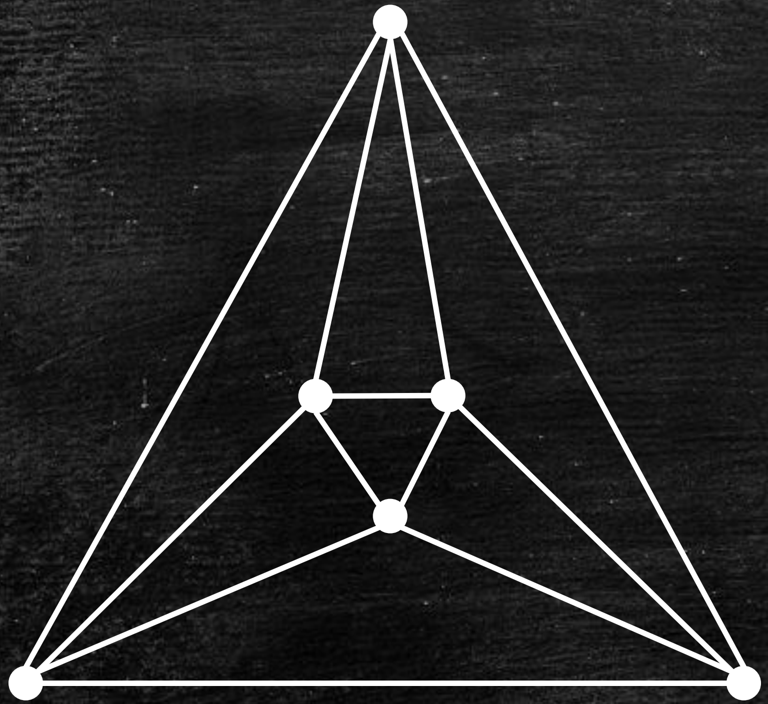
F - be the number of faces (including the single unbounded face),

Then $V - E + F = 2$.

Then $V - E + F = C + 1$.

C - is the number of connected components.

Euler's formula.



$V = 6$

$E = 12$

$F = 8$

$V - E + F = 2$

$6 - 12 + 8 = 2$

Corollary 1

Let G be any plane embedding of a connected planar graph with $V \geq 3$ vertices. Then

1. G has at most $3V - 6$ edges, and
2. This embedding has at most $2V - 4$ faces (including the unbounded one).

$V - E + F = 2$

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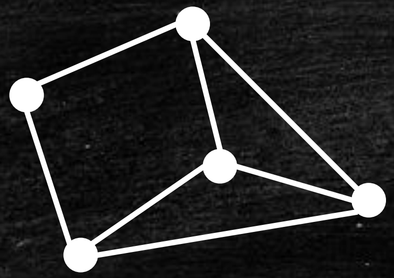

$$\left.\begin{array}{l} \sum_{i=1}^{F} e_i \leq 2E \\ \sum_{i=1}^{F} e_i \geq 3\mathrm{F} \end{array}\right\} \quad F \leq \frac{2E}{3}$$

$$V - E + F = 2$$
$$E \leq 3V - 6$$
$$F \leq 2V - 4$$

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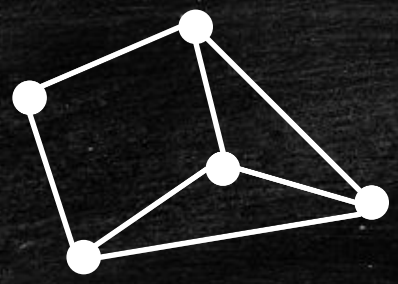
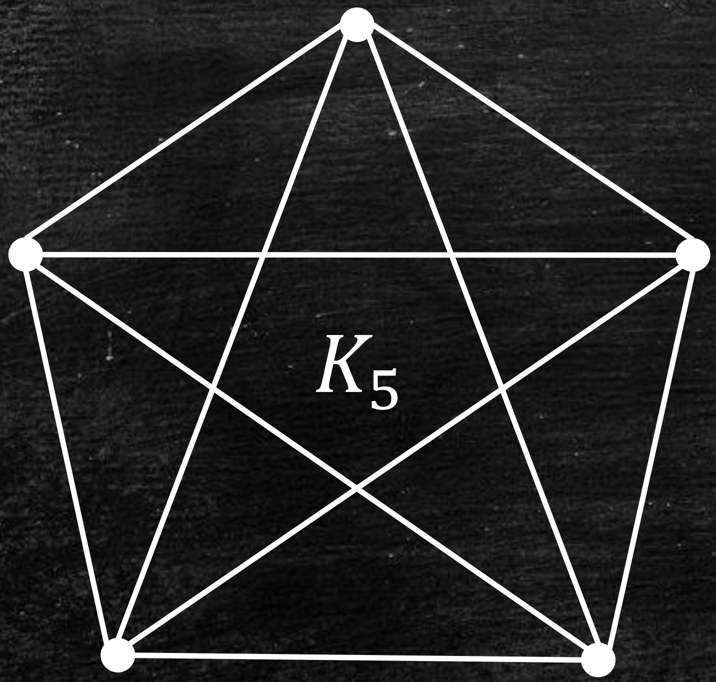


$$\left.\begin{aligned} \sum_{i=1}^{F} e_i \leq 2E \\ \sum_{i=1}^{F} e_i \geq 3\mathrm{F} \end{aligned}\right\} \quad F \leq \frac{2E}{3}$$

K_5 is not planar.

$$V - E + F = 2$$
$$E \leq 3V - 6$$
$$F \leq 2V - 4$$



$$V - E + F = 2$$
$$E \leq 3V - 6$$
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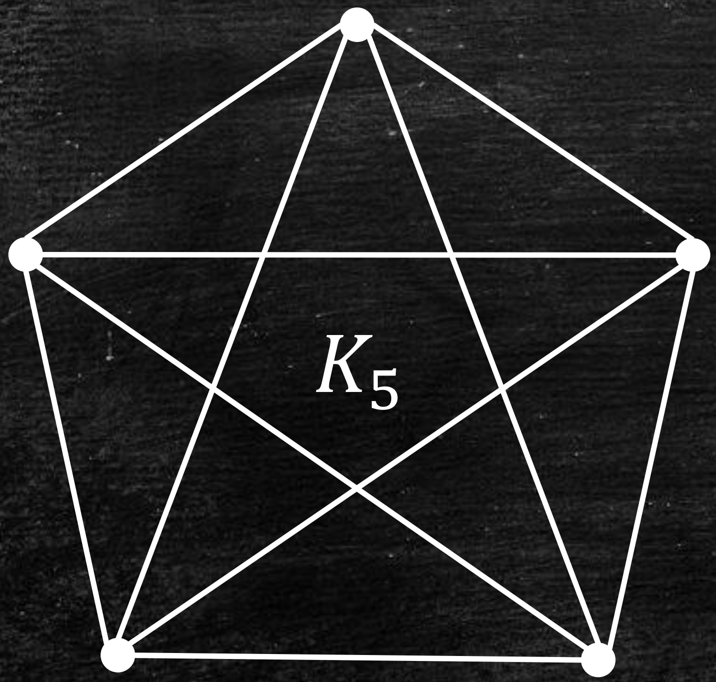
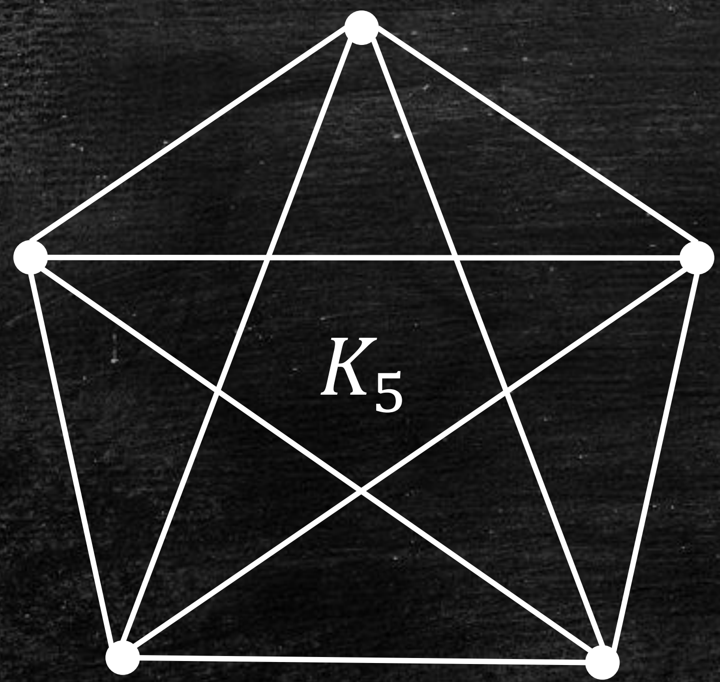


$$V = 5$$
$$E = 10$$
$$E \leq 3V - 6$$

$$V - E + F = 2$$
$$E \leq 3V - 6$$
$$F \leq 2V - 4$$

K_5 is not planar.

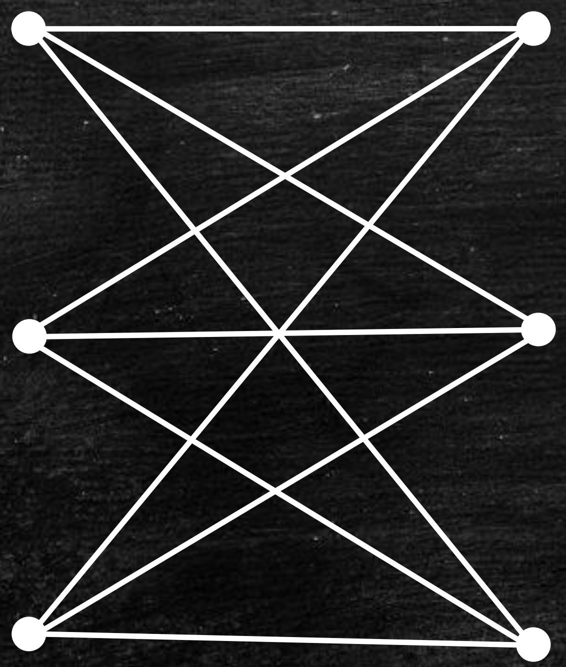


$$V = 5$$
$$E = 10$$
$$E \leq 3V - 6$$
$$10 \leq 9$$

$$V - E + F = 2$$
$$E \leq 3V - 6$$
$$F \leq 2V - 4$$

$K_{3,3}$ is not planar.

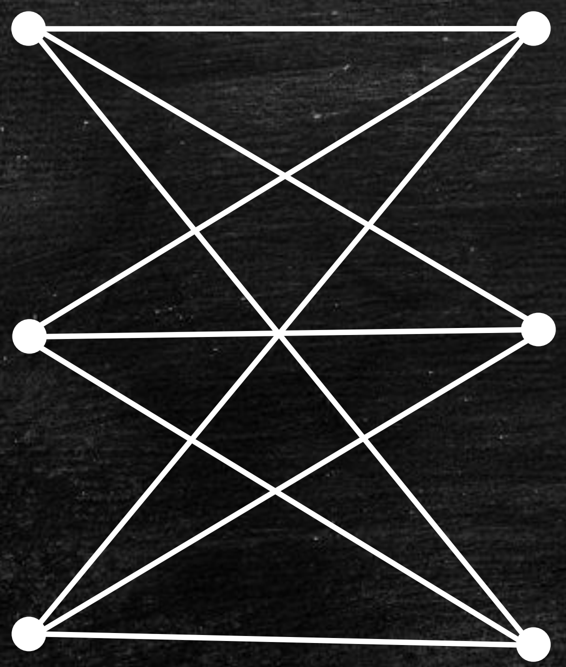
$K_{3,3}$



$$V = 6$$
$$E = 9$$
$$E \leq 3V - 6$$

$$V - E + F = 2$$
$$E \leq 3V - 6$$
$$F \leq 2V - 4$$

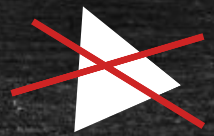
$K_{3,3}$ is not planar.



$K_{3,3}$

$$V = 6$$
$$E = 9$$
$$E \leq 3V - 6$$
$$9 \leq 12$$

Euler formula gives the necessary (but not sufficient!) condition for a graph to be planar.



$$V - E + F = 2$$
$$E \leq 3V - 6$$
$$F \leq 2V - 4$$

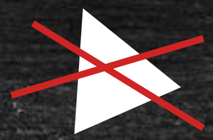
$$E \leq 2V - 4$$

Corollary 2

Let G be any plane embedding of a connected planar graph with $V \geq 4$ vertices. Assume that this embedding has no triangles, i.e. there are no cycles of length 3. Then

$$E \leq 2V - 4$$

$K_{3,3}$ is not planar.



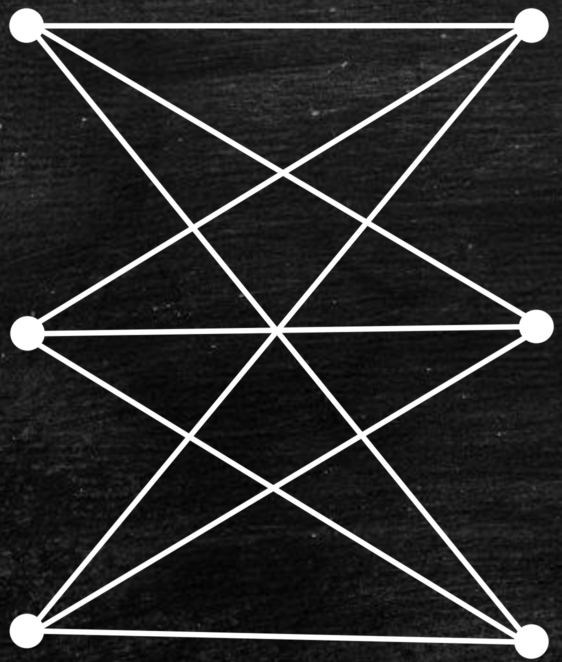
$E \le 2V - 4$

$V - E + F = 2$

$E \le 3V - 6$

$F \le 2V - 4$

$K_{3,3}$



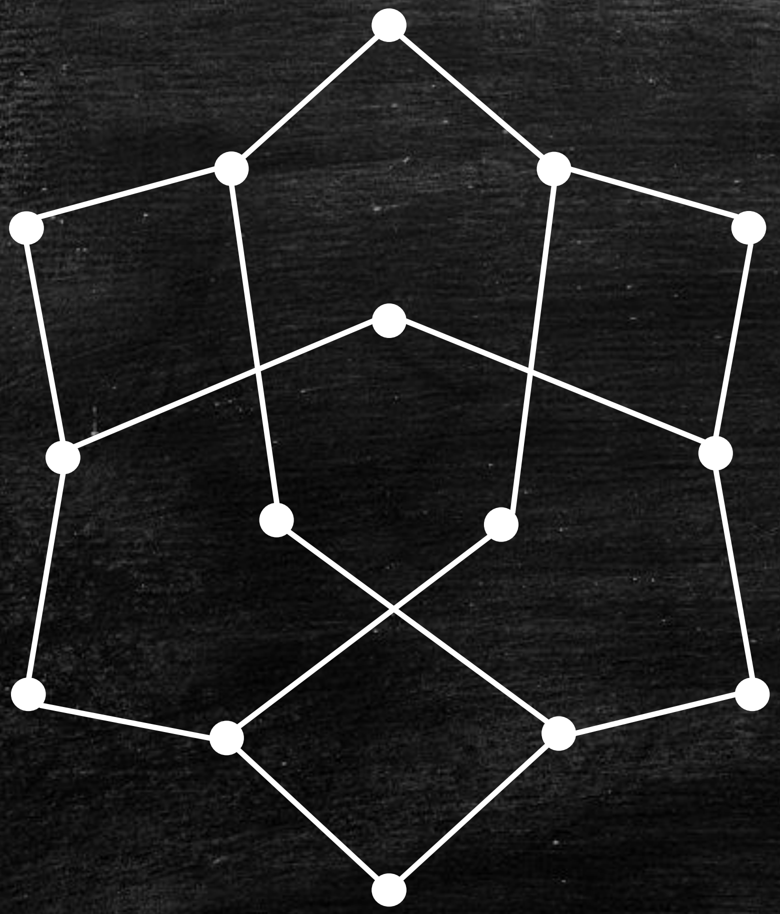
$V = 6$

$E = 9$

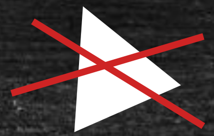
$E \le 2V - 4$

$9 \le 8$

Quiz ☺ Is the following graph planar?



Quiz ☺

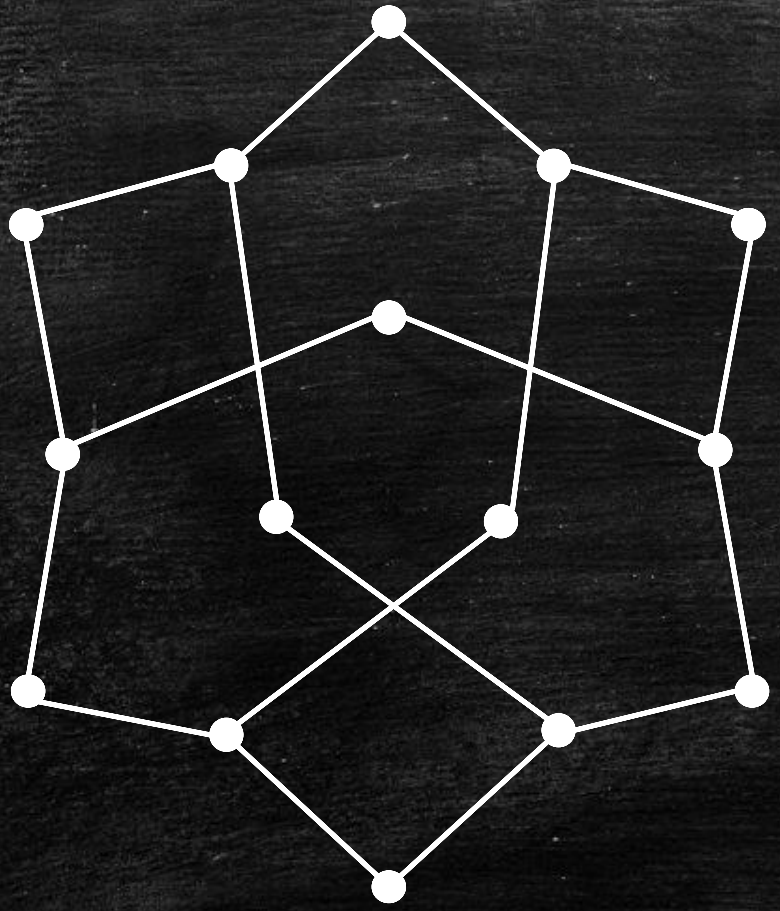


$E \leq 2V - 4$

$V - E + F = 2$

$E \leq 3V - 6$

$F \leq 2V - 4$



$V = 15$

$E = 18$

$E \leq 2V - 4$

$18 \leq 26$

What makes a graph non-planar?

- Euler's conditions are necessary but not sufficient.
- We proved that K_5 and $K_{3,3}$ are non-planar.
- Next we look at Kuratowski's and Wagner's Theorems for conditions of sufficiency.

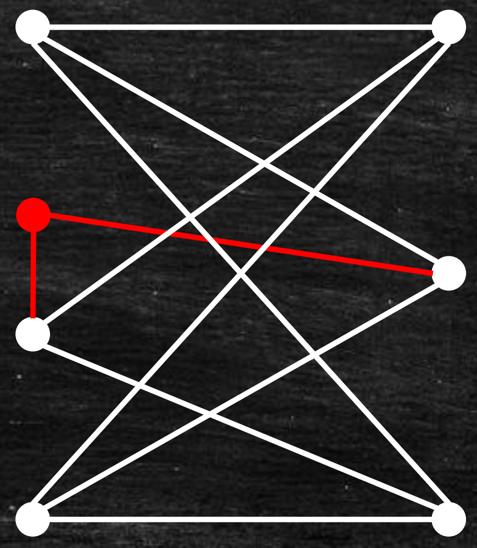
What makes a graph non-planar?

- K_5 and $K_{3,3}$ are the smallest non-planar.
- Every non-planar graph contains them, but not simply as a subgraph.
- Every non-planar graph contains a subdivision of K_5 or $K_{3,3}$.

Subdividing an edge in a planar graph does not make it non-planar.

What makes a graph non-planar?

An example of a graph which doesn't have K_5 or $K_{3,3}$ as its subgraph. However, it has a subgraph that is homeomorphic to $K_{3,3}$ and is therefore not planar.

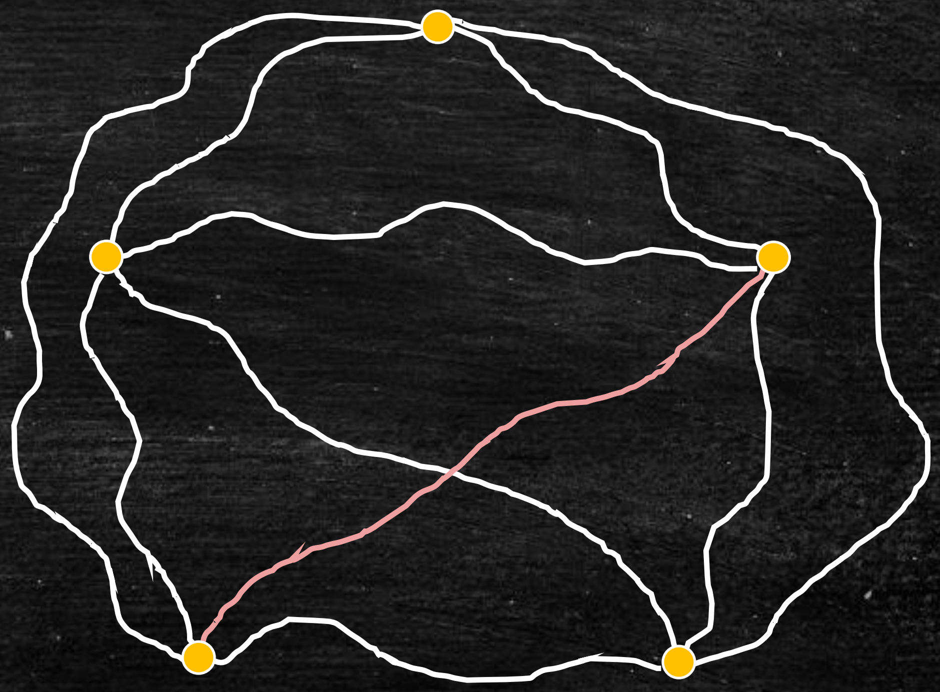


Kuratowski's Theorem.

A graph is planar <u>if and only if</u> it does not contain a subdivision of K_5 or $K_{3,3}$.

Proof:

⟸ Sufficiency immediately follows from non-planarity of K_5 and $K_{3,3}$. Any subdivision of K_5 and $K_{3,3}$ is also non-planar.



1930 by Kazimierz Kuratowski

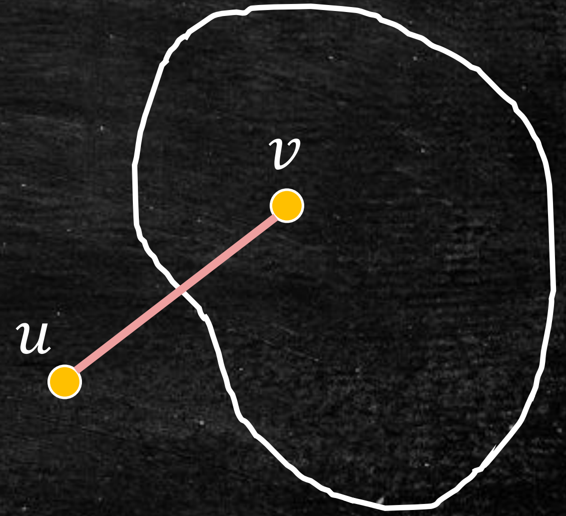
Kuratowski's Theorem.

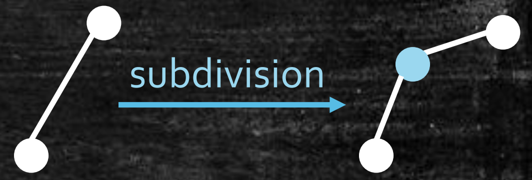
A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proof:

➡

- Suppose G is non-planar.
- Remove edges and vertices of G such that it becomes a minimal non-planar graph.
- I.e. removing any edge will make the resulting graph planar.





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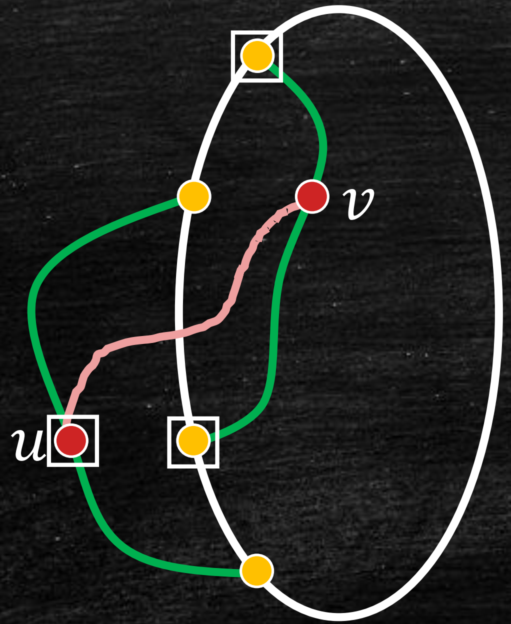

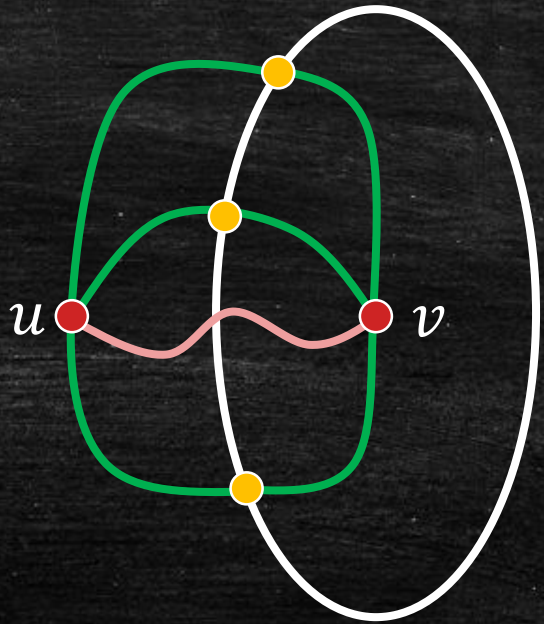

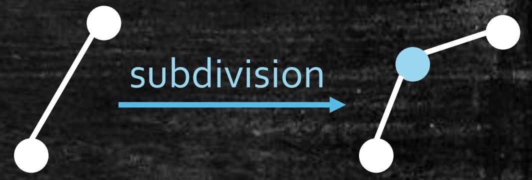

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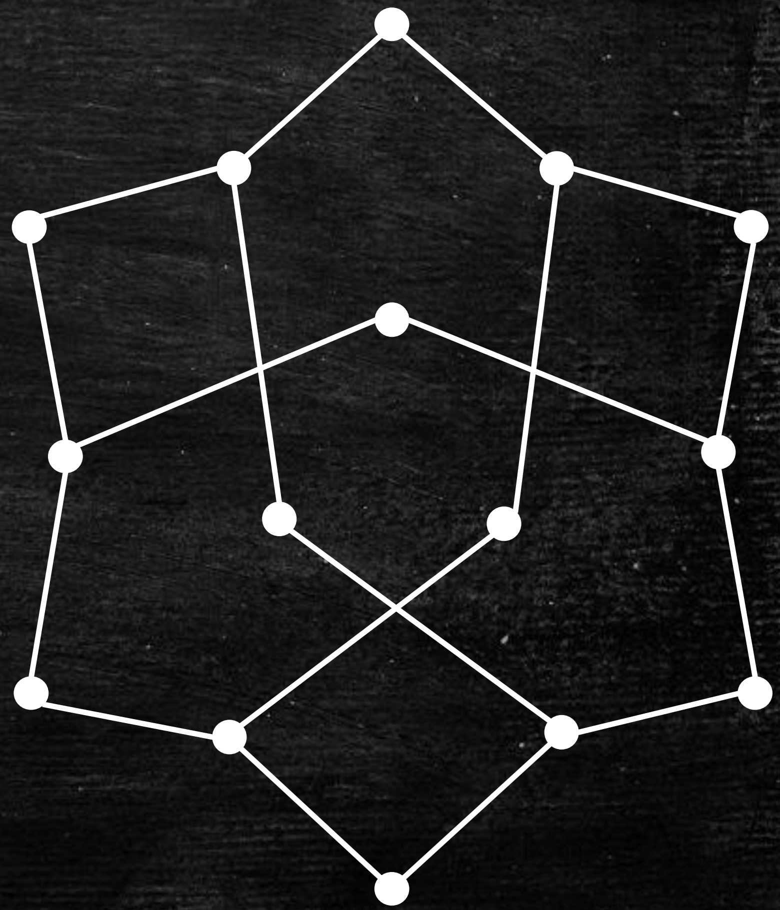

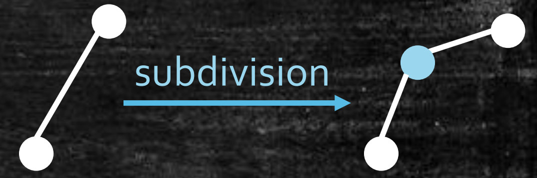

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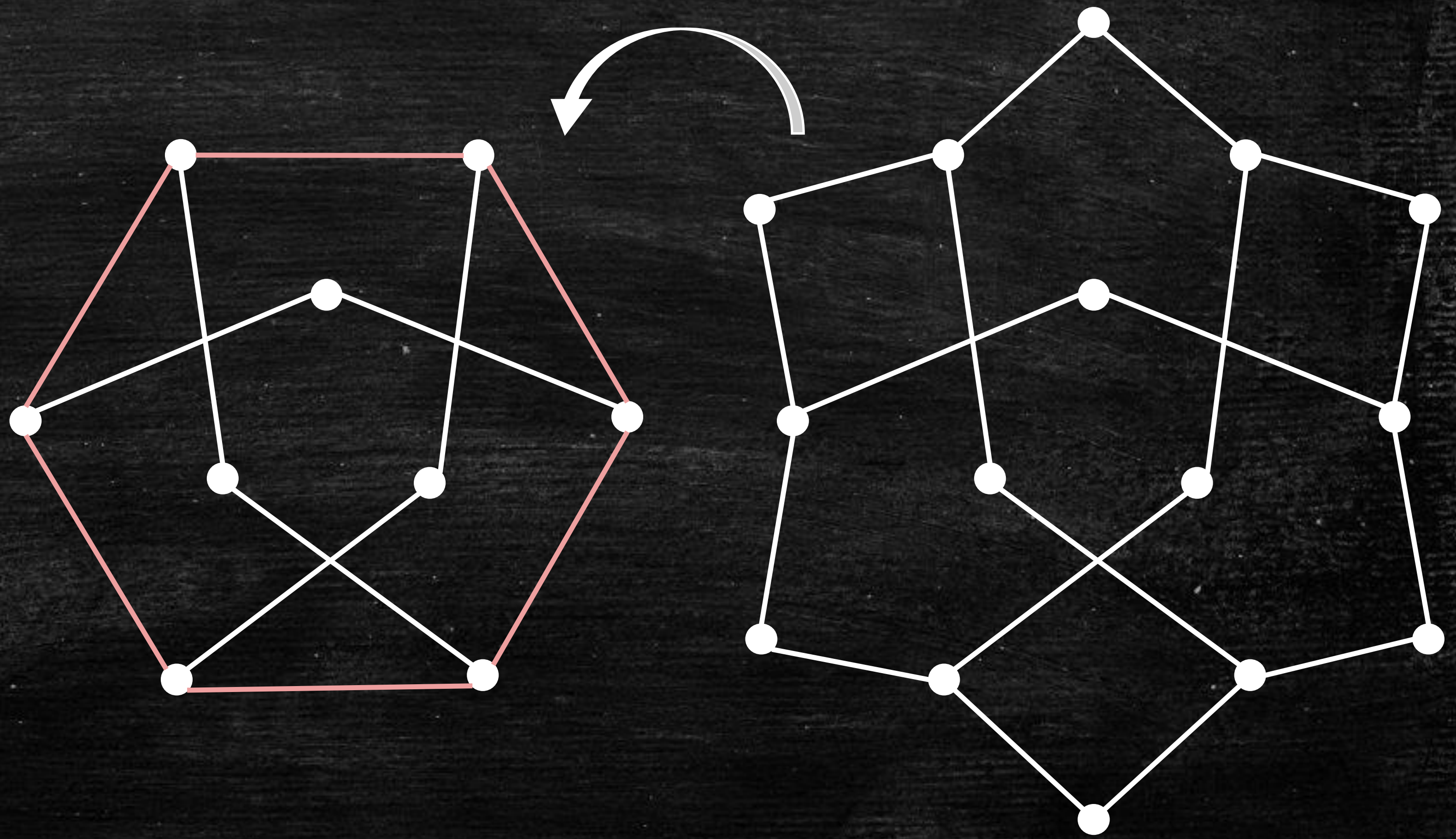

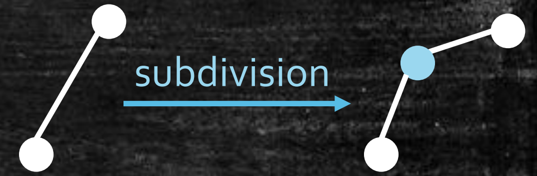

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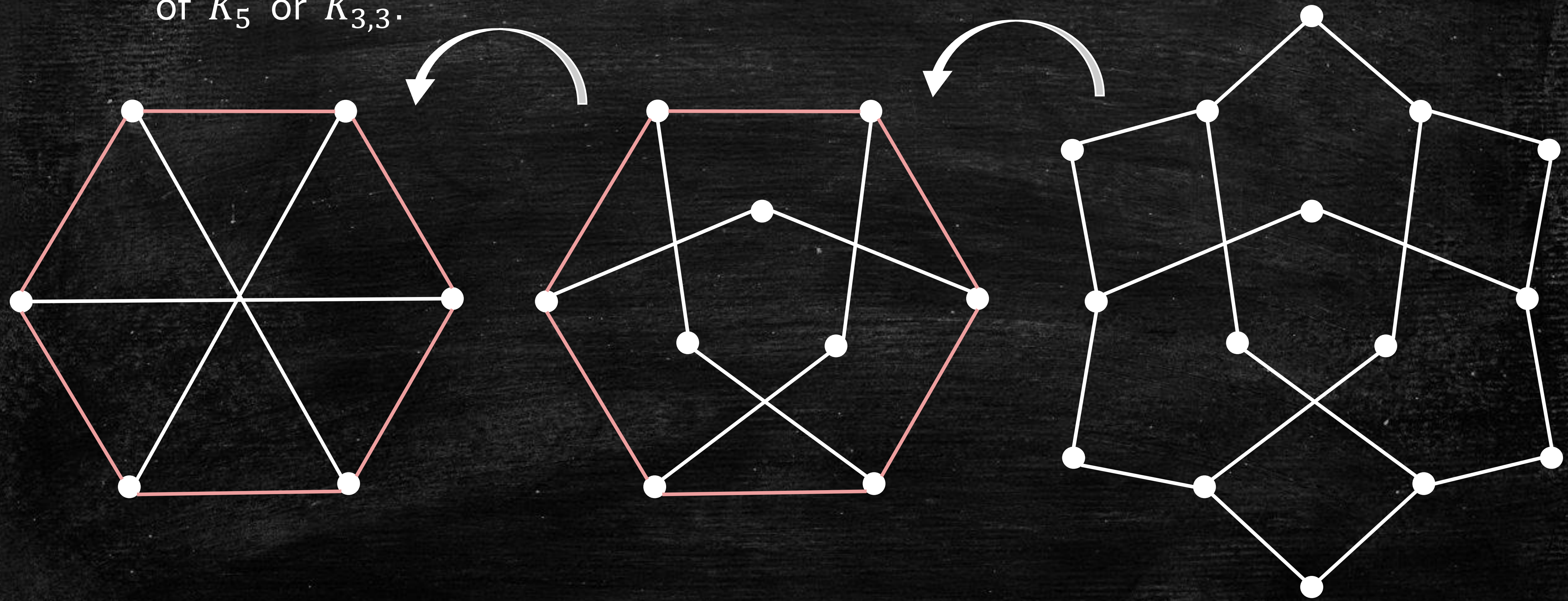
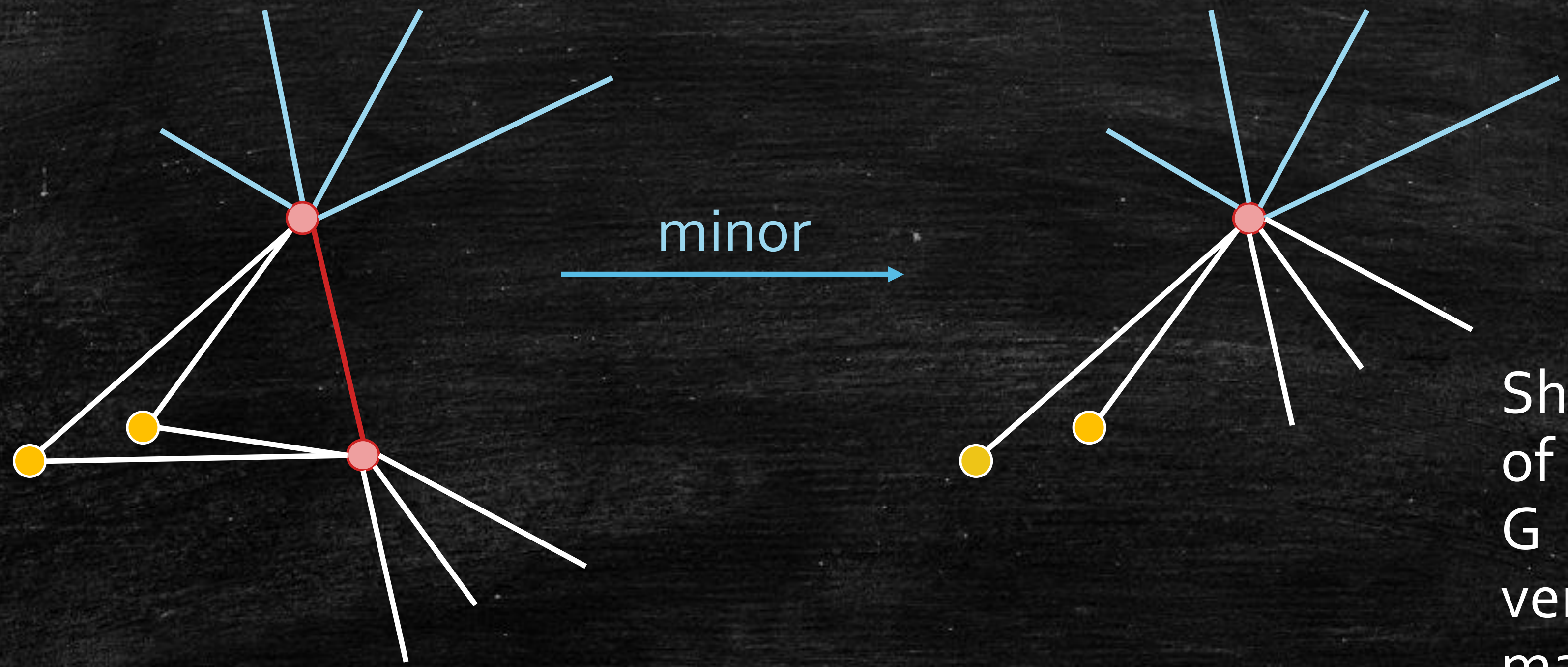


Wagner‘s Theorem.

A graph is planar if and only if it does not contain a subgraph which has K_5 or $K_{3,3}$ as a minor.

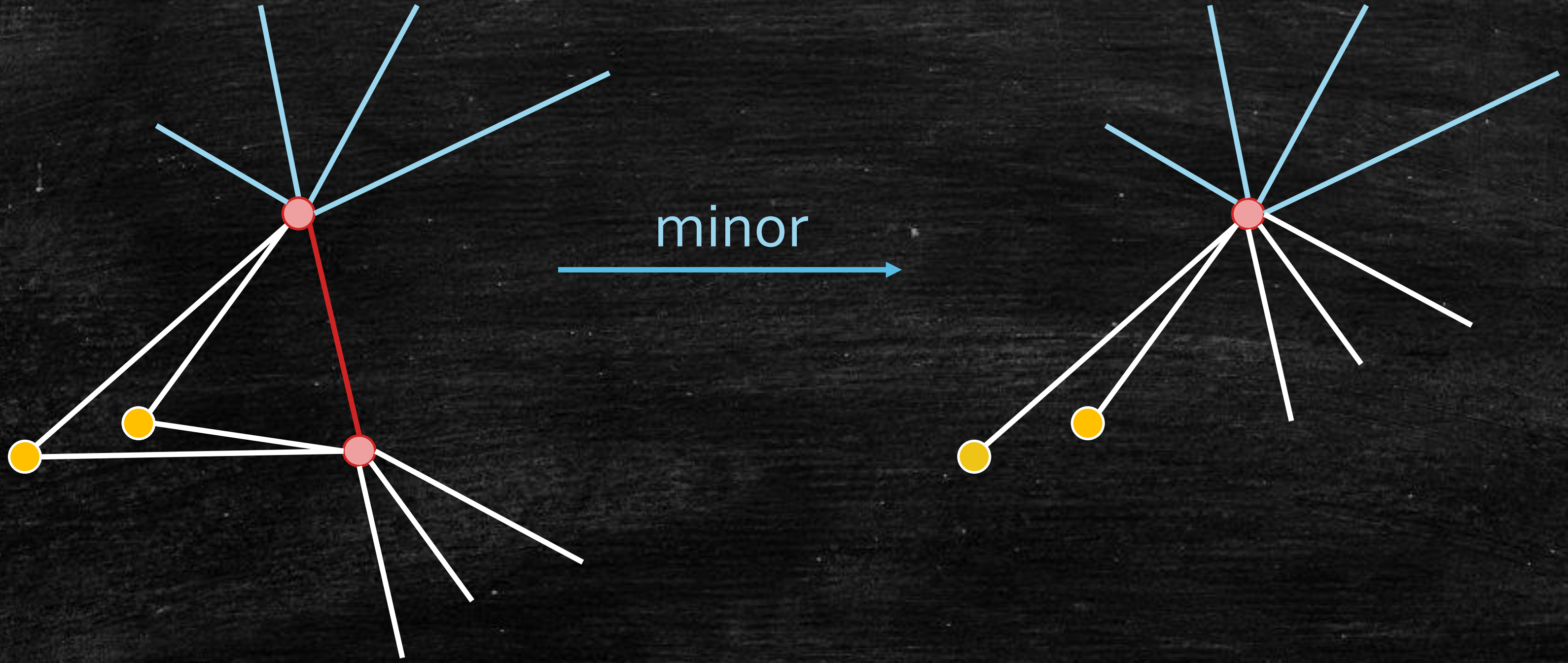


Shrinking an edge of a planar graph G to make a single vertex does not make G non-planar

1937 by Klaus Wagner

Wagner‘s Theorem.

Every graph has either a planar embedding, or a minor of one of two types: K_5 or $K_{3,3}$. It is also possible for a single graph to have both types of minor.



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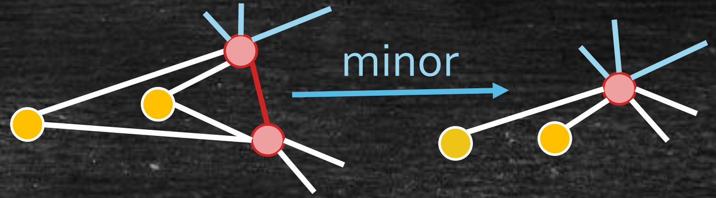
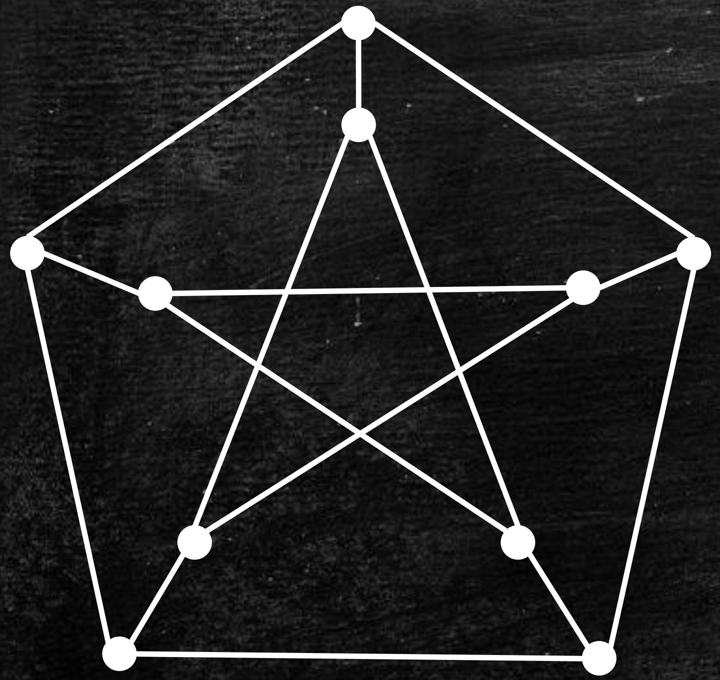
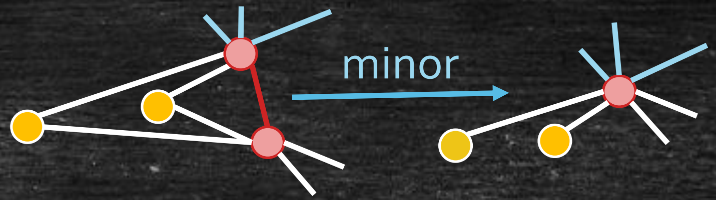


Petersen graph.

Petersen graph has both K_5 and $K_{3,3}$ as minors.



It also has a subdivision of $K_{3,3}$.



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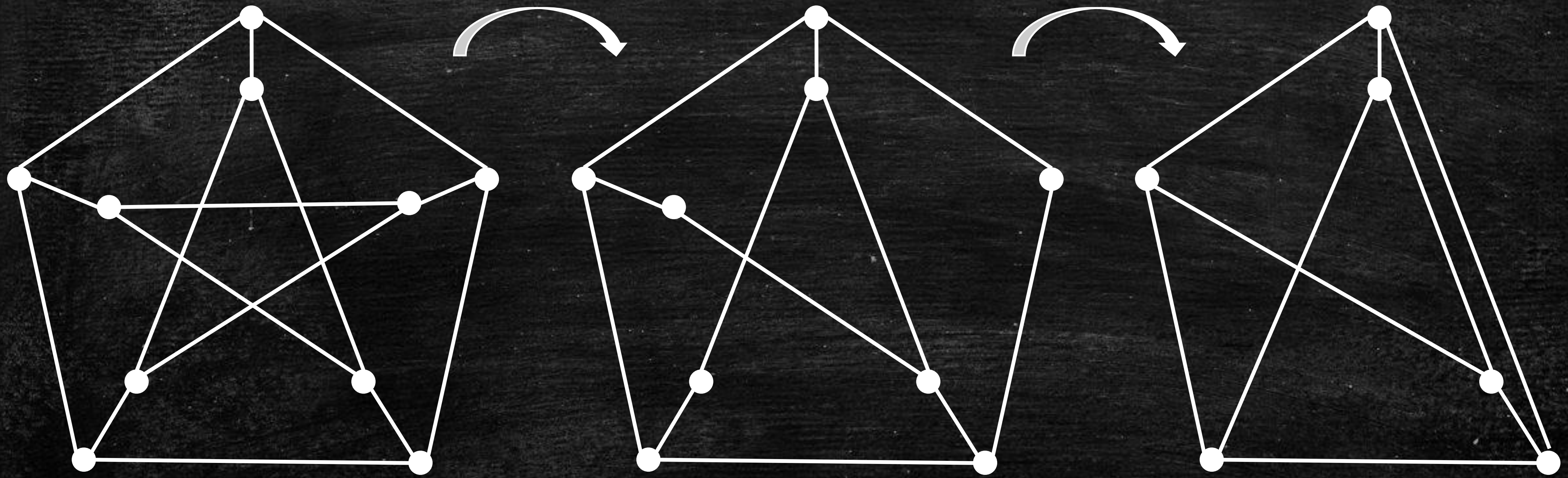

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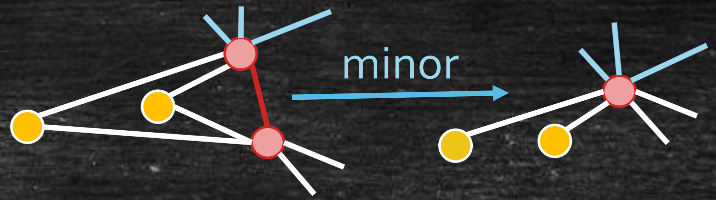

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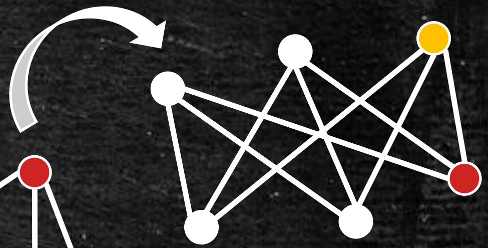

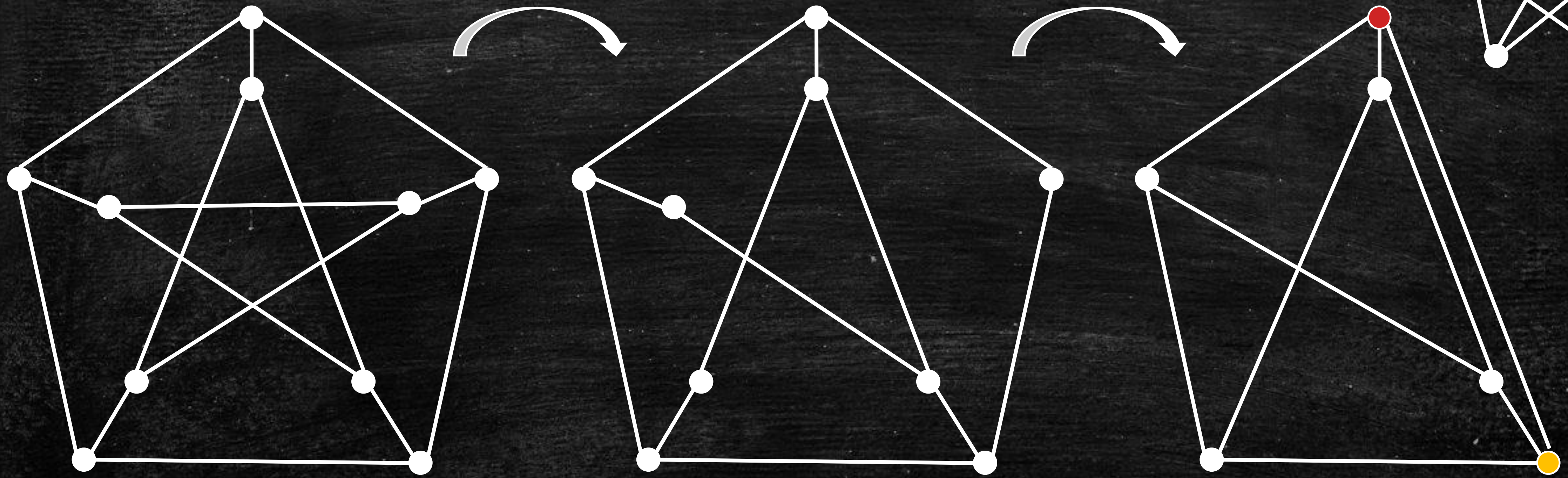

It also has a subdivision of $K_{3,3}$.

How to test planarity?

How to apply Kuratowski's theorem? Assume, you want to test a given graph G for K_5 subdivision.

- Choose 5 vertices of G.
- Check if all 5 vertices are connected by 10 distinct paths as K_5.

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Planarity testing using Wagner's Theorem:

- Choose an edge of G - there are E choices.
- Shrink it.
- If 6 vertices are remaining check for $K_{3,3}$. (if 5 - check for K_5).
- Repeat

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$O(E!)$

Planarity Algorithms.

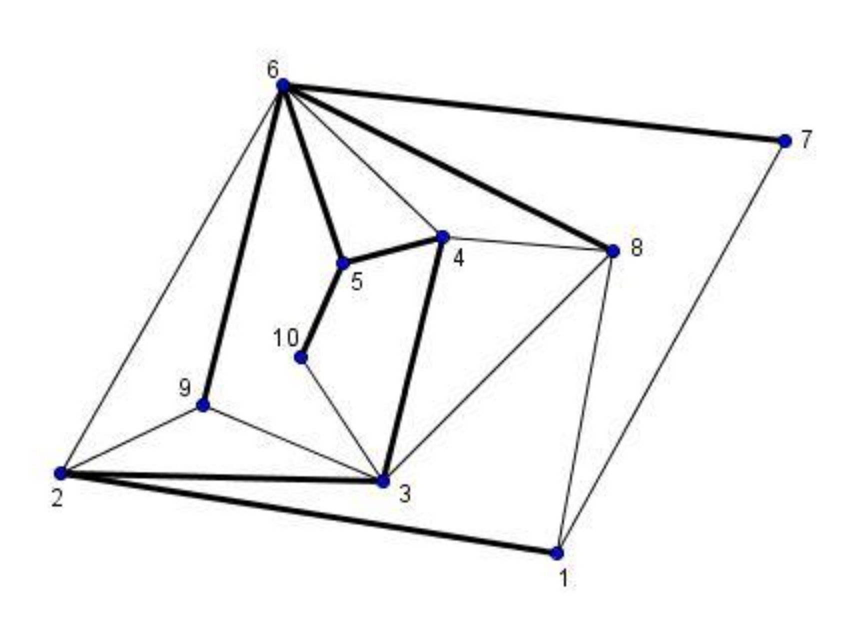
- The first polynomial-time algorithms for planarity are due to Auslander and Parter (1961), Goldstein (1963), and, independently, Bader (1964).
- Path addition method: In 1974, Hopcroft and Tarjan proposed the first linear-time planarity testing algorithm.
- Vertex addition method: due to Lempel, Even and Cederbaum (1967).
- Edge addition method: Boyer and Myrvold (2004).

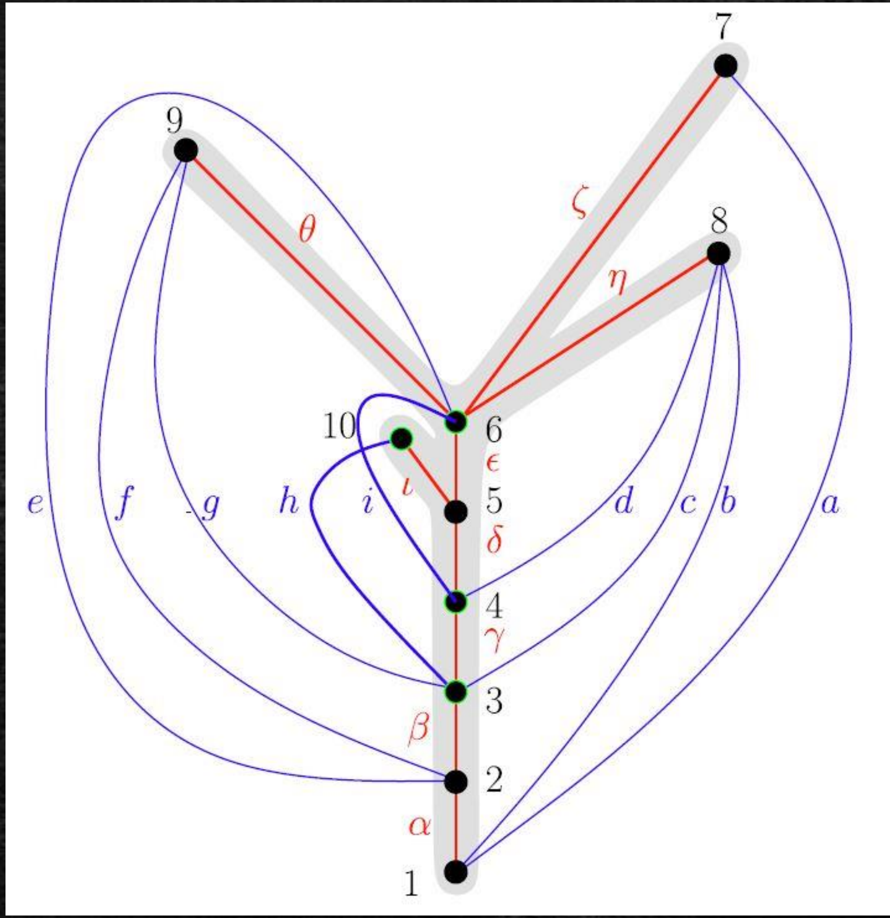
FMR Algorithm. (Left-Right algorithm)

- Due to Hubert de Fraysseix, Patrice Ossona de Mendez and Pierre Rosenstiehl. (2006)
- The fastest known algorithm.

FMR Algorithm. (Left-Right algorithm)

- The most important technique, common to almost all the algorithms, is Depth First Search.

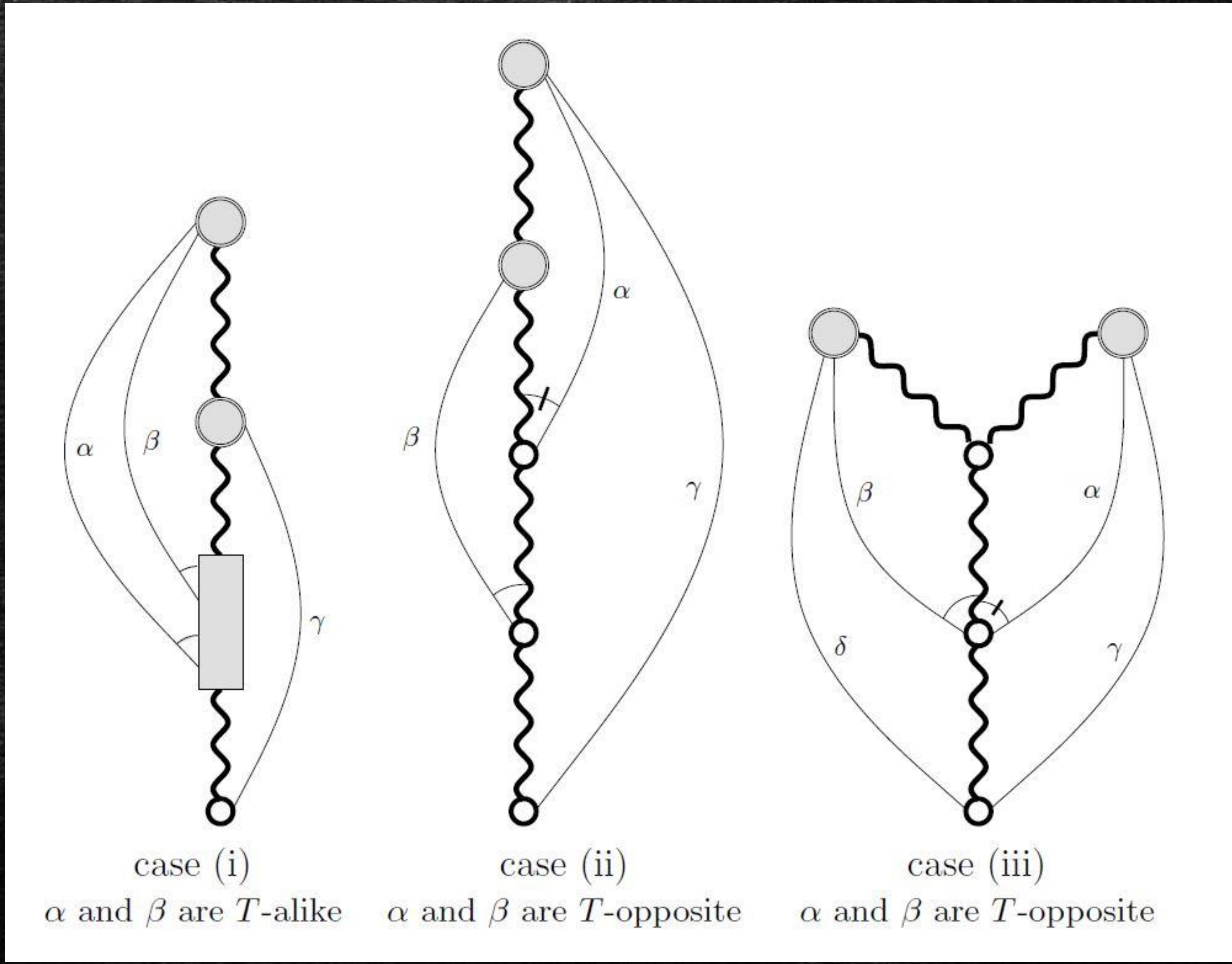




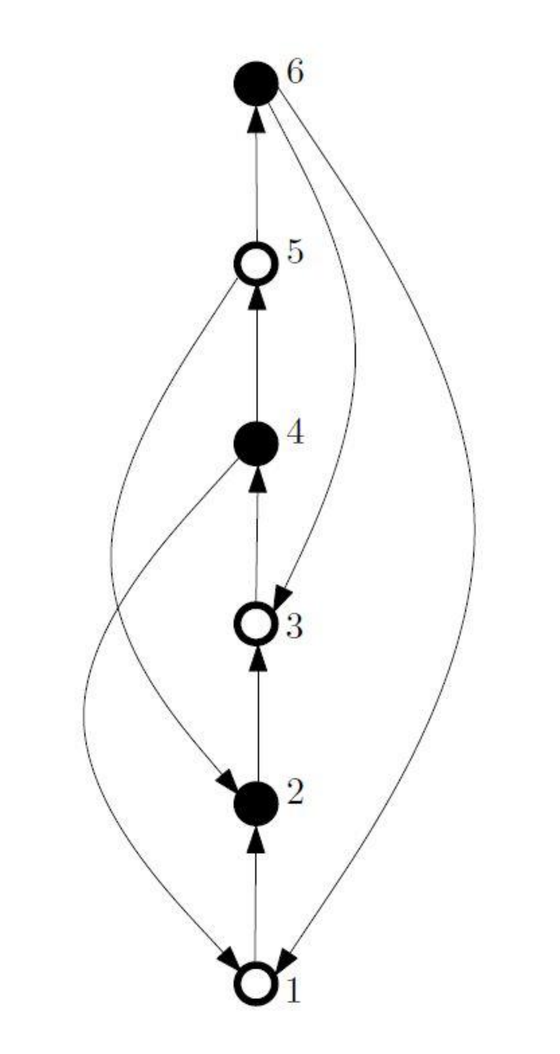
Tremaux tree
or
Palm tree

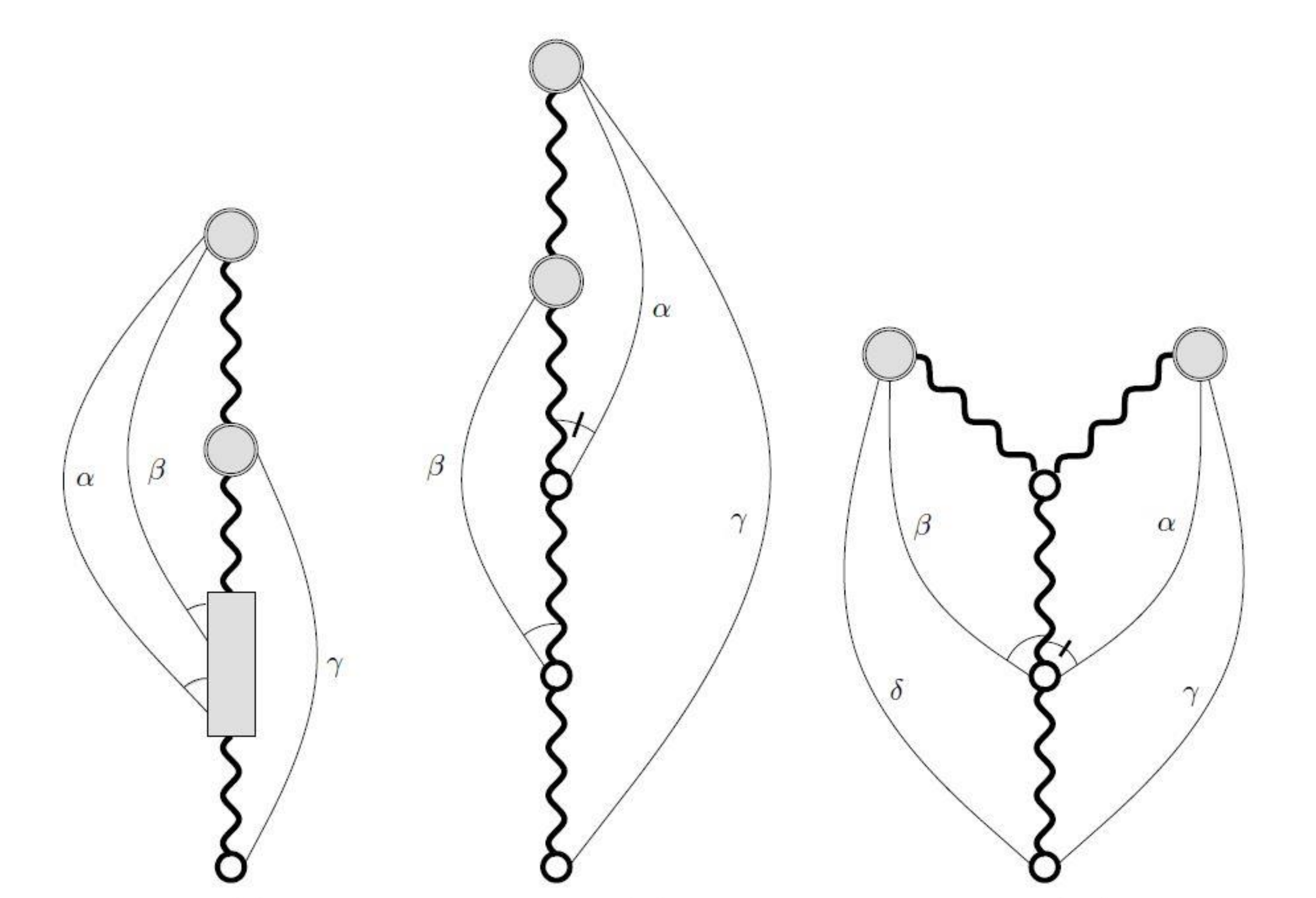
Left-Right criterion.

<u>Theorem</u>: Let G be a graph with Tremaux tree T. Then G is planar <u>**iff**</u> there exists a partition of the back-edges of G into two classes, so that any two edges belong to a same class if they are T-alike and any two edges belong to different classes if they are T-opposite.



Left-Right criterion.





case (i)
α and β are T-alike

case (ii)
α and β are T-opposite

case (iii)
α and β are T-opposite

Properties.

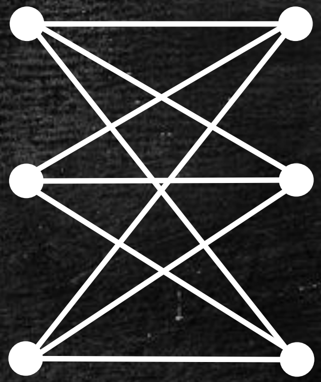
- For any connected planar graph: $E \leq 3V - 6, \;\; F \leq 2V - 4.$
- All planar graphs contain at least one vertex with degree ≤ 5.

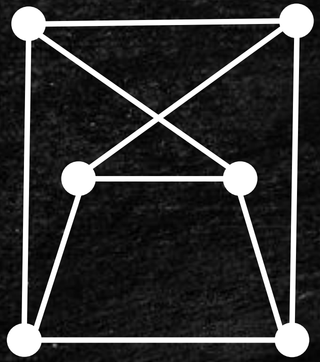
$$\sum_{i=1}^{V} d(v_i) = 2E \leq 6V - 12 < 6V$$

- Planar graphs are 4-colorable.
- Every triangle-free planar graph is 3-colorable and such a 3-coloring can be found in linear time.
- The size of a planar graph on n vertices is $O(n)$, (including faces, edges and vertices). They can be efficiently stored.

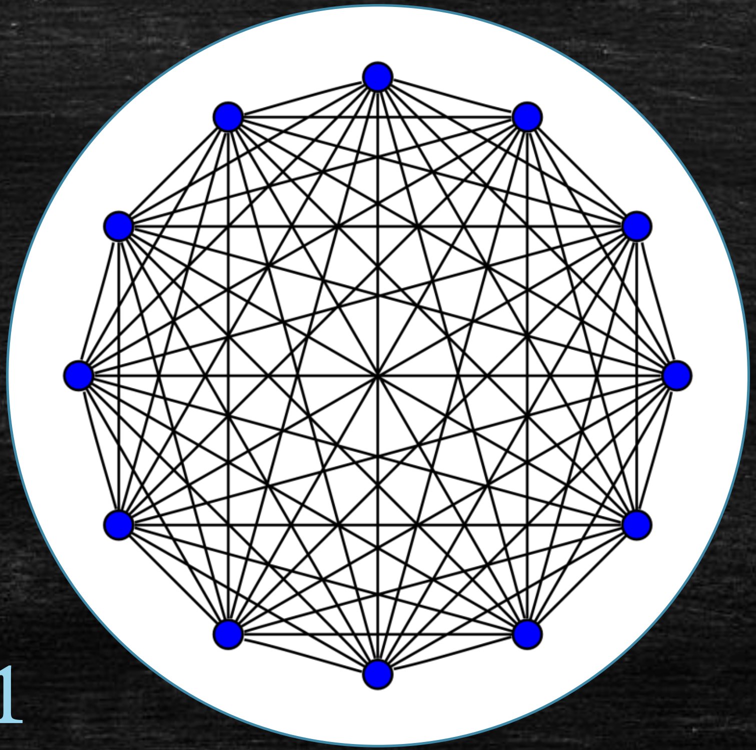
Crossing Number of G

$CR(G)$ - the minimum number of crossings over all possible embeddings of G.





$CR(K_{3,3}) = 1$



K_{12}

$E = 66$

$CR(K_{12}) = 153$

$CR(K_{29}) = ?$

Can we find a lower bound on $CR(G)$?

Given G with n vertices and m edges; select a subset of vertices of G (call it S) by picking each vertex with probability p.

$G(S)$ - the graph induces on S.

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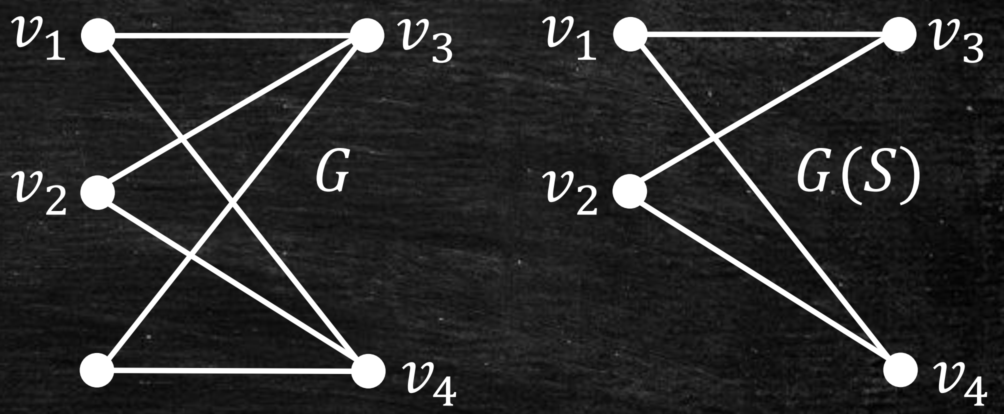


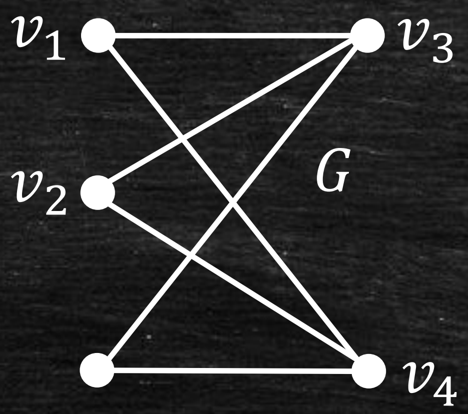
$G(S)$ - the graph induces on S.

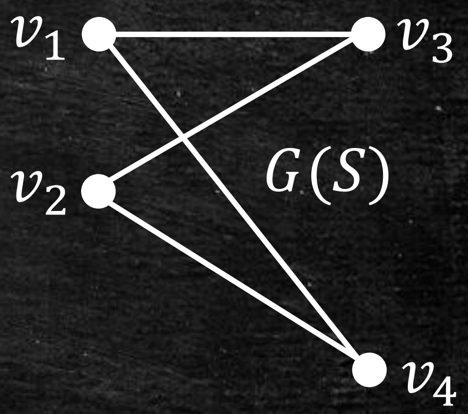
$\Pr(\overline{xy} \in G(S) | \overline{xy} \in G) = p^2$

$E\big(\# \; of \; edges \; of \; G(S)\big) = mp^2$

Can we find a lower bound on $CR(G)$?

Given G with n vertices and m edges; select a subset of vertices of G (call it S) by picking each vertex with probability p.





$G(S)$ - the graph induces on S.

$\Pr(\overline{xy} \in G(S) | \overline{xy} \in G) = p^2$

$E\big(\#\ of\ edges\ of\ G(S)\big) = mp^2$

$\Pr(crossing\ appears\ in\ G(S) | crossing\ in\ G) = p^4$

$E(\#\ of\ crossings\ in\ G(S)) = p^4 CR(G)$

$E \leq 3V - 6$

Can we find a lower bound on $CR(G)$?

$CR(G) \geq m - (3n - 6) \geq m - 3n$

$E[CR(G(S))] \geq E[m_S - 3n_s] = E[m_S] - E[3n_s]$

$p^4 CR(G) \geq mp^2 - 3pn$

$$CR(G) \geq \frac{m}{p^2} - \frac{3n}{p^3}$$

Can we find a lower bound on $CR(G)$?

$$CR(G) \geq m - (3n - 6) \geq m - 3n$$

$$E[CR(G(S))] \geq E[m_S - 3n_s] = E[m_S] - E[3n_s]$$

$$p^4 CR(G) \geq mp^2 - 3pn$$

$$CR(G) \geq \frac{m}{p^2} - \frac{3n}{p^3}$$

maximize this

$$f(p) = \frac{m}{p^2} - \frac{3n}{p^3}$$

$$f'(p) = -\frac{2m}{p^3} + \frac{9n}{p^4}$$

$$p = \frac{9n}{2m} \qquad m > \frac{9n}{2}$$

set $p = \frac{4n}{m}$

$$CR(G) \geq \frac{m}{\left(\frac{4n}{m}\right)^2} - \frac{3n}{\left(\frac{4n}{m}\right)^3} = \frac{m^3}{64n^2}$$

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