COMP 4804 Mid-term Sample Solutions

1. Run A repeatedly until it gives an answer (which, by assumption, is correct). The expected number of runs required is

$$\sum_{i=1}^{\infty} (9/10)^{i-1} \cdot \left(\tfrac{1}{10}\right) \cdot i = 10$$

So the expected running time is $10 \cdot O(n) = O(n)$.

2. Repeatedly allow B to run for up to $2 \cdot f(n)$ time units and then stop it if it hasn't yet completed. Give up after 10 iterations

By Markov's Inequality,

$$\Pr\{\text{running time of } B \geq 2 \cdot f(n)\} \leq 1/2 .$$

So $\Pr\left\{\begin{array}{l}\text{none of the 10 runs} \\ \text{finish in} \leq 2 \cdot f(n) \text{ steps}\end{array}\right\} \leq \left(\frac{1}{2}\right)^{10}$

So $\Pr\{B' \text{ is correct}\} \geq 1 - 1/2^{10}$

3. (a) $(7/8) \cdot K$

(b) Set $(1-\varepsilon)(7/8)K = K/2$

$$\iff 1-\varepsilon = \frac{1}{2} \cdot \frac{8}{7}$$

$$\iff \varepsilon = 1 - \frac{1}{2} \cdot \frac{8}{7} = 1 - \frac{4}{7} = 3/7$$

Let B be the number of correct answers (out of K runs) and we get (using Chernoff's Bounds):

$$\Pr\{B \le K/2\} = \Pr\{B \le (1-\varepsilon)\, 7/8\, K\}$$
$$\le e^{-\left(\frac{3}{7}\right)^2 \cdot \frac{7}{8} \cdot K} = e^{-\Omega(K)}$$

(c) Run C K times and output the most frequent answer. Part (b) proves the bound on the probability of correctness. The running time is immediate.

4 (a) $\Pr\{x_i \text{ is not bad}\}$

$= \Pr\{\text{color}(x_{i-1}) \neq \text{color}(x_i) \text{ and } \text{color}(x_{i+1}) \neq \text{color}(x_i)\}$

$= 1/4$

$\Rightarrow \Pr\{x_i \text{ is bad}\} = 1 - \Pr\{x_i \text{ is not bad}\} = 3/4.$

The expected number of bad elements in $\frac{3n}{4}$.

(b) $\Pr\{x_i, \dots, x_{i+t} \text{ is a bad streak}\}$

$= \Pr\{\text{color}(x_{i+1}) = \text{color}(x) \text{ and } \cdots \text{ and } \text{color}(x_{i+t}) = \text{color}(x_i)\}$

$= 1/2^t$

(c) $\Pr\{x_i \text{ starts a bad streak of lenght} \geq 2\log_2 n\}$

$= 1/2^{2\log_2 n} = 1/n^2$

By Boole's Inequality,

$\Pr\{\text{any bad streak of length} \geq t\} \leq \Pr\{x_0 \text{ starts bad streak of length} \geq t\}$
$+ \dots$
$+ \Pr\{x_{n-1} \text{ starts bad streak of length} \geq t\}$

$= n \cdot 1/2^t$

$= 1/n.$

5(a) Define $|L|$ to be the length of L and let

$$\Phi(L) = c \cdot |L|$$

2 Cases:

Case 1: The algorithm appends an element to L, so the real cost is $O(1)$ and the potential increase by c, so the amortized cost is

$$O(1+c) = O(1)$$

for any constant c.

Case 2: The real cost is $O(1+k_i)$, but the potential decrease by $c \cdot k_i$, so the amortized cost is

$$O(1+k_i) - ck_i = O(1)$$

for sufficiently large (but constant) c.

In both cases, the cost is $O(1)$