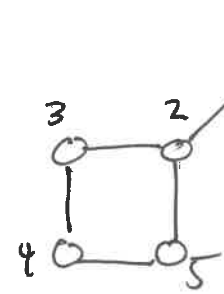
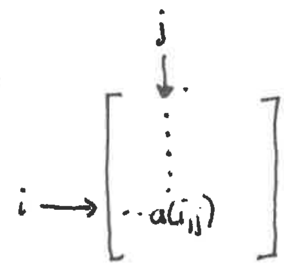
Adjacency Matrices.

For an n-vertex (directed) graph G, the adjacency matrix $A = (a(u,v) : u,v \in V(G))$ where

$$a(u,v) = \begin{cases} 1 & \text{if } uv \in E(G) \\ 0 & \text{if } uv \notin E(G). \end{cases}$$

	1	2	3	4	5
1	0	1	0	0	0
2	1	0	1	0	1
3	0	1	0	1	0
4	0	0	1	0	1
5	0	1	0	1	0





Notes: • If G has no self-loops then $\underbrace{a(v,v) = 0}_{\text{hollow.}}$ for each $v \in V(G)$.

• If G is undirected then $a(u,v) = a(v,u)$ for each $u,v \in V(G)$. (symmetric).

• If G is directed then A is not necessarily symmetric.

For not-necessarily simple graphs, we can use

$$a(u,v) = \#\text{edges from } u \text{ to } v \text{ in } G.$$

(but we won't ~~necessarily~~ really use this matrix).

Degrees.

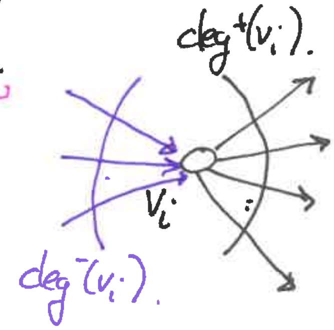
For undirected G,

$$[1\,1\,1\cdots 1]\begin{bmatrix} A \end{bmatrix} = [\deg(v_1), \deg(v_2), \dots, \deg(v_n)]$$

$$\mathbb{1}\cdot A = \mathbf{d} = \left[\sum_{i=1}^{n} a(1,i), \sum_{i=1}^{n} a(2,i), \dots, \sum_{i=1}^{n} a(n,i)\right]$$

For directed G

$$\mathbb{1}\cdot A = \underbrace{[\deg^-(v_1), \deg^-(v_2), \dots \deg^-(v_n)]}_{\text{~~out degree~~ in-degree.}}$$



$$A\cdot\mathbb{1}^T = \begin{bmatrix} A \end{bmatrix}\begin{bmatrix}1\\1\\1\\\vdots\end{bmatrix} = \begin{bmatrix}\deg^+(v_1)\\ \deg^+(v_2)\\ \vdots \\ \deg^+(v_n)\end{bmatrix} = \begin{bmatrix}\sum_{i=1}^{n} a(i,1)\\ \vdots \\ \sum_{i=1}^{n} a(i,n)\end{bmatrix}$$

~~out-degree.~~ out-degree

$$[1\,1\cdots 1]\cdot(A^T)$$

$$\mathbb{1}\cdot A^T = [\deg^-(v_1), \dots, \deg^-(v_n)]$$

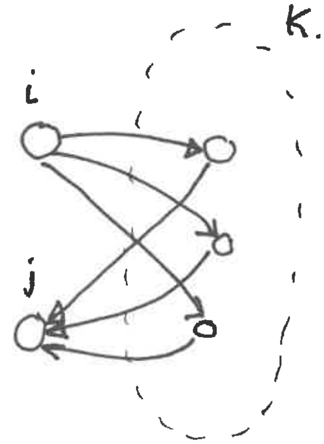
↳ Adjacency matrix of G^{rev}, the graph obtained from G by reversing the direction of each edge.

Counting Paths

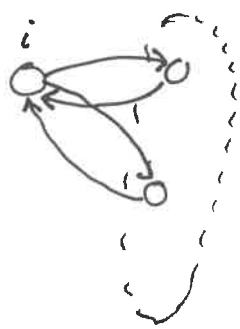
$$A^2 = \left(a^2(v,w) : v,w \in V(G)\right) \text{ where}$$

$$a^2(i,j) = \sum_{k=1}^{n} a(i,k)\cdot a(k,j)$$

= # 2-edge ~~paths~~ walks from i to j

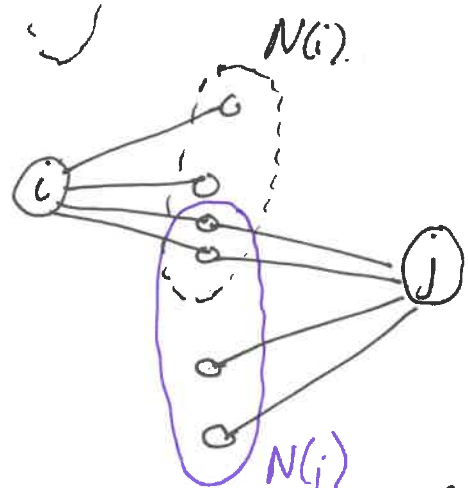


Note: Even works for $a(i,i)$.



For undirected G

$$a^2(i,j) = |N(i) \cap N(j)|$$

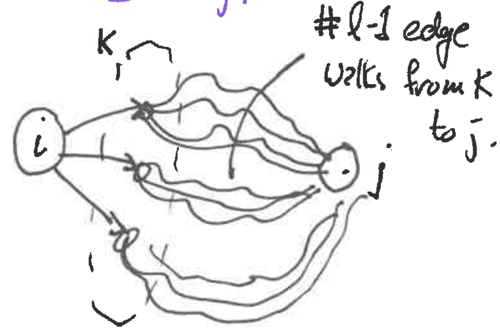


More generally

$$A^{\ell} = \left(a^{\ell}(v,w) : v,w \in V(G)\right) \text{ where}$$

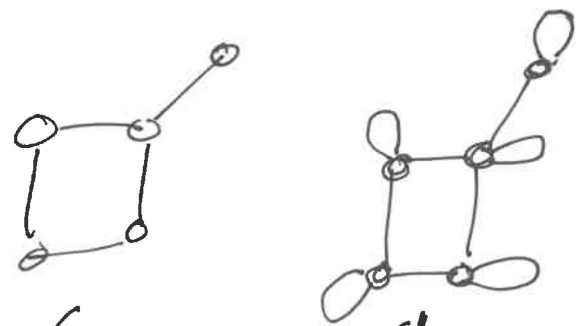
$$a^{\ell}(i,j) = \sum_{k=1}^{n} a(i,k)\cdot a^{\ell-1}(k,j)$$

= # ℓ-edge walks from i to j



$A+I$

$\begin{pmatrix} A \end{pmatrix} + \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ = adjacency matrix of (non-simple) graph G' obtained by adding a self-loop to each ~~ed~~ vertex of G.



G. G'

$(A+I)_{ij}$ = # ~~path~~ walks from i to j that use 0 or 1 edges.

$(A+I)^{\ell}_{ij}$ = # walks from i to j that use $\leq \ell$ edges.

$$(A+I)^n_{ij} = \begin{cases} 0 & \text{if } G \text{ has no path from } i \text{ to } j \\ >0 & \text{if } G \text{ has at least one path from } i \text{ to } j. \end{cases}$$

Different Kinds of Matrix Multiplication

Usual matrix multiplication is also called (sum, product)- or $(+,\times)$-multiplication:

$$(AB)_{ij} = \underbrace{\sum_{k=1}^{n}}_{\text{sum.}} \underbrace{a(i,k)\times b(k,i)}_{\text{product.}}$$

Other Kinds:

$(\max,\times)$-product $\quad (AB)_{ij} = \max_{k=1}^{n}\left(a(i,k)\times b(k,i)\right)$

Then $A^{\ell}_{ij} = \begin{cases} 1 & \text{if } G \text{ has an } \ell\text{-edge walk from } i \text{ to } j \\ 0 & \text{otherwise.} \end{cases}$

$$(A+I)^{\ell}_{ij} = \begin{cases} 1 & \text{if } \mathrm{dist}_G(i,j) \le \ell. \\ 0 & \text{otherwise.} \end{cases}$$

$(\min,+)$-product: $\quad (AB)_{ij} = \min_{k=1}^{n}\left(a(i,k)+b(k,i)\right)$

$$\left((A+I)^{*}\right)^{\ell}_{ij} = \begin{cases} \mathrm{dist}_G(i,j) & \text{if } \mathrm{dist}_G(i,j) \le \ell. \\ \infty & \text{otherwise.} \end{cases}$$

where $(A+I)^{*}_{ij} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{if } i=j \\ \infty & \text{if } i\ne j \text{ and } ij \notin E(G) \end{cases}$