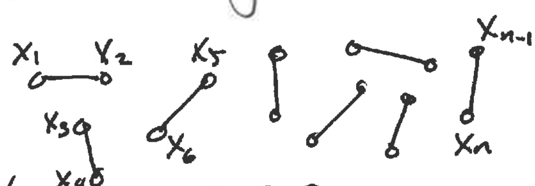
Random d-Regular Graphs.

For integer $d \geq 0$, $G_{n,d}$ is the probability space $(\Omega, \Pr)$ where Ω contains every graph G with $V(G)=[n]$ in which $\deg_G(v)=d$ for each $v \in V(G)$, and $\Pr(\omega)=\frac{1}{|\Omega|}$ for each $\omega \in \Omega$.

- $G_{n,0}$ contains one graph G with $V(G)=[n]$ and $E(G)=\emptyset$.
- $G_{n,1}$ contains all perfect matchings on $[n]$



- How many perfect matchings on $[n]$?
  - We can generate one by taking any permutation $x_1 x_2 x_3 \ldots x_n$ and grouping consecutive pairs $\{x_1,x_2\}, \{x_3,x_4\}, \{x_5,x_6\} \ldots \{x_{n-1},x_n\}$.
- Note: This only works if n is even. In fact, $G_{n,d}$ is only defined when $n \cdot d$ is even. Why? Because for $G \in \Omega$

$$dn = \sum_{v \in V(G)} \deg_G(v) = \underbrace{2 \cdot |E(G)|}_{\text{even}}$$

choose order of $x_i\, x_{i+1}$ for each $i \in \{1,3,5,\ldots n-1\}$

choose order of $\frac{n}{2}$ pairs

- For a single perfect matching G, there are $2^{n/2} \cdot (\frac{n}{2})!$ permutations that generate G
- So the number of perfect matchings on $[n]$ is

$$M(n) = \frac{n!}{2^{n/2} \cdot (\frac{n}{2})!}$$

<u>Note</u>: There is lots of freedom when generating a perfect matching:

1: Choose an unmatched vertex v using any rule you want.

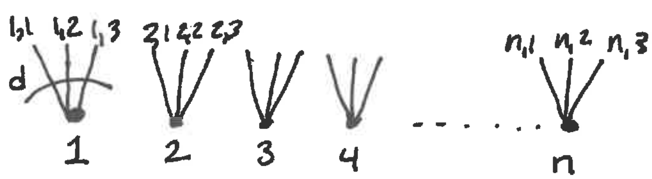
2: Choose a random unmatched vertex $w \neq v$

3: Match the pair vw.

4: Repeat until no unmatched vertices remain.

Summary: Easy to generate random ~~unmatched~~ matching $G_{n,2}$.

What about $G_{n,d}$? We use the configuration model

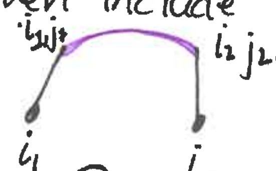


book calls these points.

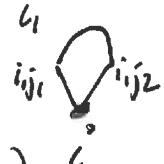
· Generate one of the $M(dn)$ matchings on the set of pairs

$\{(i,j) : i \in [n], j \in [d]\}$.

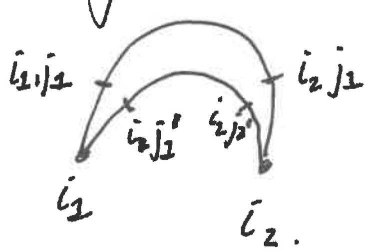
• If matching contains the pair $\{(i_1,j_1),(i_2,j_2)\}$ then include the edge $i_1 i_2$ in G.



• Issue: Can create self-loops $\{(i,j_1),(i,j_2)\}$



and multiple edges: $\{(i_1,j_1),(i_2,j_2)\}, \{(i_1,j_1'),(i_2,j_2')\}$



- So, configuration model is a probability distribution $P_{n,d} = (\Omega', Pr')$ over multigraphs (with self-loops) with vertex set $[n]$ and ordered edges.

- $\Omega' \supseteq \Omega$ since every simple graph $G \in \Omega$ is also in Ω'.

- $|\Omega'| = M(dn)$ since each element corresponds to a perfect matching on a set of size $d \cdot n$.

- For a <u>simple</u> graph G, the number of perfect matchings that generate G is $(d!)^n$

$$\Rightarrow Pr(G) = \frac{(d!)^n}{M(dn)}.$$

- So we can generate a ~~uniformly~~ graph $G \sim G_{n,d}$ by generating an element from $P_{n,d}$ and rejecting it if it is not simple.

- $Pr'(G \text{ is simple}) \xrightarrow{n \to \infty} e^{-(d^2-1)/4}$ [Book states this without proof, so the proof must be hard.]

- The ~~expected~~ number of samples we have to take before getting a simple G is a geometric random variable with expected value $e^{(d^2-1)/4}$.

- This is very useable for $d \le 6$ and still useable for $d \le 10$

If we can ~~choose~~ show that $\Pr(P_{n,d} \text{ has property } X) \xrightarrow{n\to\infty} 1$ then $\Pr(G_{n,d} \text{ has property } X) \longrightarrow 1$ since

$$\Pr(G_{n,d} \text{ has property } X) = 1 - \Pr(G_{n,d} \text{ doesn't have } X)$$

$$= 1 - \Pr(P_{n,d} \text{ doesn't have } X \mid P_{n,d} \text{ is simple})$$

$$\Rightarrow 1 - \frac{\Pr(P_{n,d} \text{ doesn't have } X \text{ and } P_{n,d} \text{ is simple})}{\Pr(P_{n,d} \text{ is simple})}.$$

$$\xrightarrow{n\to\infty} 1 - e^{(d^2-1)/4} \cdot 0 = 1 - 0 = 1.$$

Summary: If we can show that $P_{n,d}$ has property X with high probability, then $G_{n,d}$ has property X with high prob.

$G_{n,2}$ a collection of cycles.

- How many components does $G_{n,2}$ have?
- How many components does $P_{n,2}$ have.

- Choose a vertex v_0 then choose neighbour v_1 of v_0
  - then choose neighbour v_2 of v_1

    ⋮
  - stop when $v_i = v_1$.
  - repeat this until ~~until~~ we have n edges.

$$2(n-i)+1.$$
$$\|$$

- When choosing the i^{th} edge, we have $2n-(2(i-1))-1$ options.

$2(i-1)$ of these are used by $i-1$ existing edges.
1 is used by current vertex.

- Exactly one choice completes ~~closes~~ a cycle.
- $Pr(\text{step } i \text{ creates a cycle}) = \dfrac{1}{2(n-i)+1}$
- $$\mathbb{E}(\#\text{cycles}) = \sum_{i=1}^{n} Pr(\text{step } i \text{ creates a cycle}) = \sum_{i=1}^{n} \frac{1}{2(n-i)+1}$$
$$= \frac{1}{2} \cdot \sum_{i=1}^{n} \frac{1}{n-i+1} \approx \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i} = \frac{1}{2} \cdot H_n = \frac{1}{2} \ln n + 1.$$

$$Pr\left(\tfrac{1}{2}\ln n - c \le \#\text{cycles} \le \tfrac{1}{2}\ln n + c\right) \xrightarrow{c \to \infty} 1.$$

Because events E_i := "step i closes a cycle" are mutually independent.

$G_{n,3}$: Random 3-regular graph.

Results: $\Pr(G_{n,3} \text{ is connected}) \to 1$.

$\Pr(G_{n,3} \text{ is an expander}) \to 1$.

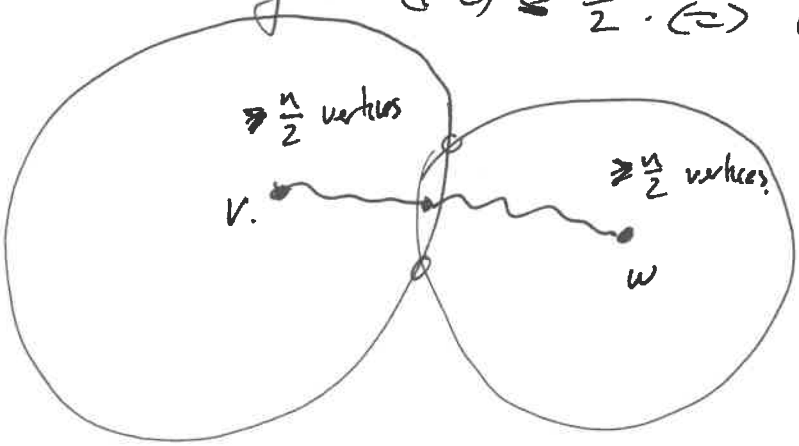
For any $S \subseteq V(G)$ with $|S| \le \frac{n}{2}$, $|N_{G_{n,3}}(S)| \geq \varepsilon \cdot |S|$.

for some fixed $\varepsilon > 0$.

Expander implies $\operatorname{diam}(G_{n,3}) \in O(\log n)$ with prob $\to 1$.

For any v, $|\{w \in V(G_{n,3}) : dist(v,w) \le i\}| \geq (1+\varepsilon)^i$,

as long as $(1+\varepsilon)^{i-1} \le \frac{n}{2}$. $\iff i \le \dfrac{\log(\frac{n}{2})}{\log(1+\varepsilon)} + 1$.



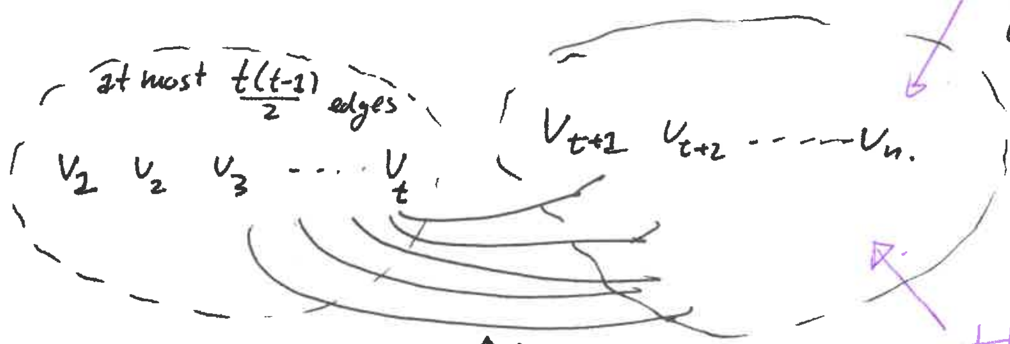
Random Graphs with a Given Degree Sequence.

Let $\underline{d} = (d_1, d_2, d_3, \ldots, d_n)$ be a sequence of integers (~~$\sum_i d_i$ is even~~).

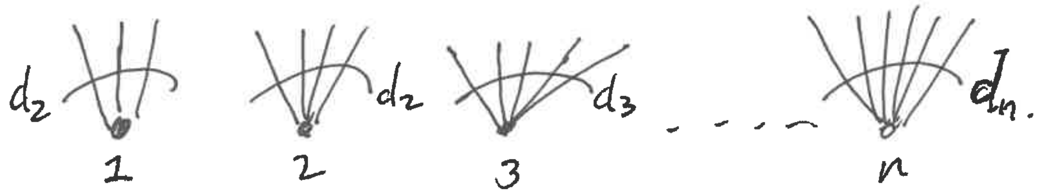
$d_1 \le d_2 \le d_3 \le \ldots d_n$.

$\underline{d}$ is graphic if $\sum_{i=1}^{n} d_i$ is even and

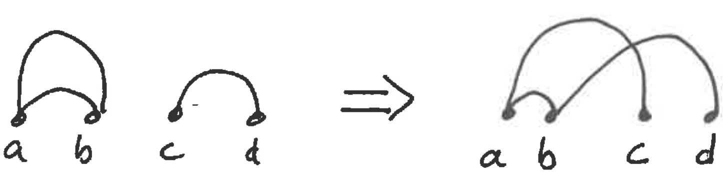
$$\sum_{i=1}^{t} d_i \overset{\text{minus}}{-} t(t-1) \le \sum_{i=t+1}^{n} d_i$$



- $G_{n,\underline{d}}$ is uniform probability distribution over graphs G with $V(G) = [n]$ and where $\deg_G(i) = d_i$ for each $i \in [n]$.
- Can also generate these using configuration model



- Configuration model might give non-simple graph (loops or multiple edges)
- This might be ok (then it doesn't even matter if $\underline{d}$ is graphic).
- Options:
  - <u>Erased configuration model</u>: Just remove any loop and all but one occurrence of each parallel edge.
  - Changes the degree sequence, but maybe not too much.
  - Rejection Method: Not very practical. For many interesting degree sequences, the probability that the resulting graph is simple is small. Requires running the method for too long
  - Switching algorithm: While there is a loop or multiple edge, choose a random edge and switch endpoints



  - Gives correct degree sequence
  - If $\sum_{i=1}^{n} d_i = O(n)$ and $\sum_{i=1}^{n} d_i^2 = O(n)$ then this produces simple graph after one iteration and the resulting simple graph is almost uniformly distributed.