Problem 1. In class, we have seen that if $A$ and $B$ are regular languages over an alphabet $\Sigma$, then so is the language $A \cup B$. Which of the following languages are guaranteed to be regular as well? Prove that your answer is correct.

- $\overline{A} = \Sigma^* \setminus A = \{w \in \Sigma^* : w \notin A\}$;
- $A \cap B = \{w \in \Sigma^* : w \in A \land w \in B\}$;
- $A \setminus B = \{w \in \Sigma^* : w \in A \setminus B\}$; and
- $A \oplus B = \{w \in \Sigma^* : w \text{ is in exactly one of } A \text{ and } B\}$. 


Problem 2. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and consider the DFA $M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$. Is it always true that $L(M') = \overline{L(M)} = \Sigma^* \setminus L(M)$?

What if one took a NFA $N = (Q, \Sigma, \delta, q_0, F)$ and built a NFA $N' = (Q, \Sigma, \delta, q_0, Q \setminus F)$? Would it always be true that $L(N') = \overline{L(N)}$?

Prove that your answers are correct.
Problem 3. Consider the following scheme to describe chess positions. We use the letters $P$, $B$, $N$, $R$, $Q$, and $K$ to denote, respectively, white pawns, bishops, knights, rooks, queens and kings; their black counterparts are denoted by the corresponding lower-case letter; empty board cells are denoted by the underscore character (_); and a chess position is denoted by the string of characters that is obtained by traversing the board in row-major order. For instance, the position depicted below is denoted by the string

$$
\_\_r\_\_\_b\_\_\_\_R\_\_\_\_p\_\_\_n\_\_\_\_p\_\_\_B\_\_\_q\_\_\_K\_\_\_P\_\_\_\_\_\_\_Q.
$$

Consider some standard rule set of chess and the language of all strings over the alphabet $\{P, B, N, R, Q, p, n, r, q, k, _\}$ that, under our established scheme, represent a legal chess position from which, if White were to start the game and play perfectly, White would be assured victory. Is this language regular? Prove that your answer is correct.
**Problem 4.** One way to define a balanced sequence of parentheses is as a string over the alphabet $\Sigma = \{(,\)\} such that: its total number of ( characters equals its total number of ) characters; and in each of its prefixes, there are at least as many ( characters as there are ) characters.

The debt of a prefix of a balanced sequence of parentheses is the number of ( characters in the prefix minus the number of ) characters in it. The depth of a balanced sequence of parentheses is the maximum debt among all its prefixes.

For a given $d \in \mathbb{N}$, we can define the language $P_d$ of all balanced sequences of parentheses having depth at most $d$. We can also define the language

$$P = \bigcup_{d \in \mathbb{N}} P_d = P_0 \cup P_1 \cup \cdots$$

of all balanced sequences of parentheses.

Later in the course we shall see that $P$ is not a regular language. This will be shown in a systematic manner, but it is possible to prove this fact using only material covered in class. For now this is left as a challenging exercise but is not part of the assignment.

In this problem you must answer the following question and prove that your answer is correct. Which values of $d \in \mathbb{N}$ are such that $P_d$ is a regular language?