Problem 1. [10 points] Let

\[ \Sigma = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \{0, 1\} \right\}. \]

A string \( w \in \Sigma^* \) can then be understood as encoding two natural numbers in binary notation. More precisely, if

\[ w = \prod_{i=n-1}^{0} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} \cdots \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \]

then the numbers encoded are

\[ \sum_{i=0}^{n-1} 2^i a_i \text{ and } \sum_{i=0}^{n-1} 2^i b_i. \]

Consider the language

\[ \text{LESS} = \left\{ \prod_{i=n-1}^{0} \begin{bmatrix} a_i \\ b_i \end{bmatrix} : \sum_{i=0}^{n-1} 2^i a_i < \sum_{i=0}^{n-1} 2^i b_i \right\}. \]

Draw a diagram describing a DFA that recognizes LESS.
Problem 2. [30 points] Let $\Sigma$ be an alphabet. A proper prefix of a string $w \in \Sigma^*$ is a string $x \in \Sigma^*$ for which there exists a non-empty string $y \in \Sigma^* \setminus \{\varepsilon\}$ such that $xy = w$. Similarly, a proper suffix of a string $w \in \Sigma^*$ is a string $y \in \Sigma^*$ for which there is a non-empty string $x \in \Sigma^* \setminus \{\varepsilon\}$ such that $xy = w$.

Given an arbitrary DFA $M = (Q, \Sigma, \delta, q_0, F)$, construct a DFA $M' = (Q', \Sigma, \delta', q'_0, F')$ by formally defining $Q'$, $\delta'$, $q'_0$ and $F'$ in such a way that

$$L(M') = \{w \in L(M) : \text{no proper prefix of } w \text{ is in } L(M)\}.$$  

Construct also a DFA $M'' = (Q'', \Sigma, \delta'', q''_0, F'')$ by formally defining $Q''$, $\delta''$, $q''_0$ and $F''$ in such a way that

$$L(M'') = \{w \in L(M) : w \text{ is not a proper prefix of any string in } L(M)\}.$$  

Let $A$ be a regular language. Is it always true that $\{w \in A : \text{no proper suffix of } w \text{ is in } A\}$ is regular? Prove that your answer is correct.
Problem 3. [20 points] Let $A$ be a language over an alphabet $\Sigma$. Define $A_+ = \{xay : x, y \in \Sigma^*, a \in \Sigma$ and $xy \in A\}$, i.e., $A_+$ is the language of strings that can be obtained by inserting a character into a string in $A$. Is $A_+$ guaranteed to be regular whenever $A$ is regular? Prove that your answer is correct.
Problem 4. [10 points] Show a context-free grammar describing the language \( \{0^a1^b0^{a+b} : a, b \in \mathbb{N}\} \).
Problem 5. [15 points] Prove that the language \( \{ x\#y : x, y \in \{0, 1\}^* \text{ and } x \neq y \} \) over the alphabet \( \{0, 1, \#\} \) is not regular and show a context-free grammar describing it.
Problem 6. [15 points] Prove that the language \( \{xy : x, y \in \{0, 1\}^*, |x| = |y| \text{ but } x \neq y\} \) is not regular and show a context-free grammar describing it.