

# A Center Transversal Theorem for Hyperplanes and Applications to Graph Drawing

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## Abstract

Motivated by an open problem from graph drawing, we study several partitioning problems for line and hyperplane arrangements. We prove a ham-sandwich cut theorem: given two sets of  $n$  lines in  $\mathbb{R}^2$ , there is a line  $\ell$  such that in both line sets, for both halfplanes delimited by  $\ell$ , there are  $\sqrt{n}$  lines which pairwise intersect in that halfplane, and this bound is tight; a centerpoint theorem: for any set of  $n$  lines there is a point such that for any halfplane containing that point there are  $\sqrt{n/3}$  of the lines which pairwise intersect in that halfplane. We generalize those results in higher dimension and obtain a center transversal theorem, a same-type lemma, and a positive portion Erdős-Szekeres theorem for hyperplane arrangements. This is done by formulating a generalization of the center transversal theorem which applies to set functions that are much more general than measures. Back to Graph Drawing (and in the plane), we completely solve the open problem that motivated our search: there is no set of  $n$  labelled lines that are universal for all  $n$ -vertex labelled planar graphs. As a side note, we prove that every set of  $n$  (unlabelled) lines is universal for all  $n$ -vertex (unlabelled) planar graphs.

## 1 Introduction

Consider a mapping of the vertices of a graph to distinct points in the plane and represent each edge by the closed line segment between its endpoints. Such a graph representation is a (*straight-line*) *drawing* if the only vertices that each edge intersects are its own endpoints. A *crossing* in a drawing is a pair of edges that intersect at some point other than a common endpoint. A drawing is *crossing-free* if it has no crossings.

One main focus in graph drawing is finding methods to produce drawings or crossing-free drawings for a given graph with various restrictions on the position of the vertices of the graph in the plane. For instance, there is plethora of work where vertices are required to be placed on integer grid points or on parallel lines in 2 or 3-dimensions.

Given a set  $R$  of  $n$  regions in the plane and an  $n$ -vertex graph  $G$ , consider a class of graph drawing problems where  $G$  needs to be drawn crossing-free by placing each vertex of  $G$  in one region of  $R$ . If such a drawing exists, then  $R$  is said to *support*  $G$ . The problems studied in the literature distinguish between two scenarios: in one, each vertex of the graph is prescribed its specific region (that is, the vertices and the regions are labelled); in the other, each vertex is free to be assigned to any of the  $n$  regions (that is, the vertices are unlabelled).

When regions are points in the plane, Rosenstiehl and Tarjan [RT86] asked if there exists a set of  $n$  points that support all  $n$ -vertex unlabelled planar graphs. This question is answered in

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the negative by De Fraysseix [dFPP88]. On the contrary, every set of  $n$  points in general position supports all  $n$ -vertex unlabelled outerplanar graphs, as proved by Gritzmann *et al.*[GMPP91] and Bose [Bos02]. If the drawings are not restricted to be straight-line, then every set of labelled points supports every labelled planar graph, as shown by [PW01]. However  $\Omega(n)$  bends per edge may be necessary in any such crossing-free drawing.

When regions are labelled lines in the plane, Estrella-Balderrama [EBFK09] showed that for every  $n \geq 6$ , there is no set of  $n$  *parallel* lines in the plane that support all labelled  $n$ -vertex planar graphs. The authors moreover characterized a (sub)class of  $n$ -vertex planar graphs that are supported by every set of  $n$ -parallel lines, for every labelling of the graphs in the class. That class is mainly comprised of several special families of trees. Dujmović *et al.*[DEK<sup>+</sup>10] showed that no set of  $n$  lines that *all* intersect in one common point supports all  $n$ -vertex labelled planar graphs. Moreover, they show that for every  $n$  large enough, there is a set of  $n$  lines in general position that does not support all labelled  $n$ -vertex planar graphs. They leave as the main open problem the question of whether, for every  $n$  large enough, there exists a set of  $n$  lines in the plane that support all labelled  $n$ -vertex planar graphs. In Section 5, as our main graph drawing result, we answer that question in the negative. As a side note, we show that in the unlabelled case, every set of  $n$ -lines supports every  $n$ -vertex unlabelled planar graph. The result illustrates the sharp contrast with the labelled case.

While the positive result is proved using little of the geometry in the arrangement, the non-existence of universal line sets requires to extract some (bad) substructure from any line arrangement. This prompted us to study several structural and partitioning problems for line and hyperplane arrangements.

**Hyperplane arrangements.** Partitioning problems are central to our understanding of discrete and computational geometry, and while many works have focused on partitioning point sets, probability distributions or measures, much less is understood for sets of lines in  $\mathbb{R}^2$  or hyperplanes in  $\mathbb{R}^d$ . This is partially due to the fact that a line (or a hyperplane) being infinite can't be contained in any bounded region, or even in a halfplane (except if the boundary of the halfplane is parallel to the given line). Previous works (such as cuttings [CF90, Mat90] or equipartitions [LS03]) have focused on identifying, and bounding the number of lines/hyperplanes intersecting a set of regions. Others [CSSS89] on partitioning the vertices of the arrangements rather than the lines themselves. Those results have found numerous applications. Our graph drawing problem motivates a different approach.

An arrangement  $L$  of  $n$  lines in  $\mathbb{R}^2$  is composed of vertices  $V(L)$  (all pairwise intersections between lines of  $L$ ), edges connecting these vertices, and half-lines. If we omit the half-lines, we are left with a finite graph which can be contained in a bounded region of the plane, in particular, it is contained in the convex hull  $CH(V(L))$  of the vertices of the arrangement. Therefore, a natural way of evaluating the portion of an arrangement contained in a given convex region  $C$  is to find the largest subset  $L'$  of lines of  $L$  such that the arrangement of  $L'$  (without the half-lines) is contained in  $C$ , or equivalently, such that all pairwise intersections of lines in  $L'$  lie in  $C$ .

It is not hard to show that, in any arrangement of  $n$  lines, a line  $\ell$  can be found such that both closed halfplanes bounded by  $\ell$  contain  $\sqrt{n}$  lines which pairwise intersect in that halfplane. This provides the analogue of a bisecting line for point sets. In Section 3.1, we show that any two line arrangements can be bisected simultaneously in this manner, thus proving a ham-sandwich theorem for line arrangements. We also prove a centerpoint theorem: for any arrangement of  $n$  lines, there is a point  $q$  such that for any halfplane containing  $q$ , there are  $\sqrt{n/3}$  lines of the arrangement that pairwise intersect in that halfplane. In Section 3.2 we generalize these notions

to higher dimensions and prove a center transversal theorem: for any  $k$  and  $d$ , there is a growing function  $Q$  such that for any  $k$  sets  $A_1, \dots, A_k$  of hyperplanes in  $\mathbb{R}^d$ , there is a  $(k-1)$ -flat  $\pi$  such that for any halfspace  $h$  containing  $\pi$  there is a subset  $A'_i$  of  $Q(|A_i|)$  hyperplanes from each set  $A_i$  such that any  $d$  hyperplanes of  $A'_i$  intersect in  $h$ . The bound  $Q$  we find is related to Ramsey numbers for hypergraphs.

Ham-sandwich theorems have a number of natural consequences. In Section 2 we show a same-type lemma for hyperplane arrangements: informally, for any  $k$  arrangements  $A_1, \dots, A_k$  of hyperplanes in general position (no  $d+1$  share a point) and that are large enough, we can find a large subset of hyperplanes  $A'_i$  from each set  $A_i$  such that the convex hulls  $CH(A'_i)$  of the vertices in the arrangements  $A'_i$  are well-separated, that is, no hyperplane hits  $d+1$  of them. In the plane, we also show a positive portion Erdős-Szekeres theorem: for any integers  $k$  and  $c$  there is an integer  $N$  such that any set of  $N$  lines in general position contains  $k$  subsets  $A_1, \dots, A_k$  of  $c$  lines each such that the vertices of each arrangement  $A_i$  can be separated from those of all the others by a line.

All the results above would be relatively easy to prove if the set function we were computing – the maximum subset of hyperplanes that have all  $d$ -wise intersections in a given region – was a measure. Unfortunately it is not. However, in Section 2, we identify basic properties much weaker than those of measures which, if satisfied by a set function, guarantee a central-transversal theorem to be true.

## 2 Center transversal theorem

The center transversal theorem is a generalization of both the ham-sandwich cut theorem, and the centerpoint theorem discovered independently by Dol'nikov [Dol92], and Živaljević and Vrećica [ŽV90]. The version of Dol'nikov is defined for a class of set functions that is more general than measures. Let  $\mathcal{H}$  be the set of all open halfspaces in  $\mathbb{R}^d$  and let  $\mathcal{G}$  be a family of subsets of  $\mathbb{R}^d$  closed under union operations and that contains  $\mathcal{H}$ . A charge  $\mu$  is a finite set function that is defined for all set  $X \in \mathcal{G}$ , and that is monotone ( $\mu(X) \leq \mu(Y)$  whenever  $X \subseteq Y$ ) and subadditive ( $\mu(X \cup Y) \leq \mu(X) + \mu(Y)$ ). A charge  $\mu$  is concentrated on a set  $X$  if for all halfspace  $h \in \mathcal{H}$  s.t.  $h \cap X = \emptyset$ ,  $\mu(h) = 0$ . Dol'nikov shows<sup>1</sup>:

**Theorem 1** (Center transversal theorem [Dol92]). *For arbitrary  $k$  charges  $\mu_i$ ,  $i = 1, \dots, k$ , defined on  $\mathcal{G}$  and concentrated on bounded sets, there exists a  $(k-1)$ -flat  $\pi$  such that*

$$\mu_i(h) \geq \frac{\mu_i(\mathbb{R}^d)}{d-k+2}, i = 1, \dots, k,$$

for all open halfspace  $h \in \mathcal{H}$  containing  $\pi$ .

A careful reading of the proof of this theorem reveals that its statement can be generalized, and the assumptions on  $\mu_i$  weakened. We first notice that the subadditive property is only used in the proof for taking the union of a finite number of halfspaces from  $\mathcal{H}$ . Therefore, define  $\mu$  to be  $\mathcal{H}$ -subadditive if

$$\mu(\cup_{h \in H} h) \leq \sum_{h \in H} \mu(h)$$

for any finite set  $H \subset \mathcal{H}$  of halfspaces.

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<sup>1</sup>Dol'nikov actually shows a slightly more general theorem that allows for non-concentrated charges. For the sake of simplicity we only discuss the simplified version even though our generalizations extend to the stronger original result.

Next, notice that in order for the proof to go through, the set function  $\mu$  need not be real-valued. Recall [Bou07] that a totally ordered unital magma  $(M, \oplus, \leq, e)$  is a totally ordered set  $M$  endowed with a binary operator  $\oplus$  such that  $M$  is closed under  $\oplus$  operations,  $\oplus$  has neutral element  $e$  (i.e.,  $x \oplus e = x = e \oplus x$ ) and is monotone (i.e.,  $a \oplus c \leq b \oplus c$  and  $c \oplus a \leq c \oplus b$  whenever  $a \leq b$ ). Further, for all  $x \in M$  and  $c \in \mathbb{N}$ , define the  $c^{\text{th}}$  multiple of  $x$  as  $cx := \oplus^c x := \underbrace{x \oplus (x \oplus (\dots \oplus x) \dots)}_{c \text{ times}}$ .

Then, it suffices that  $\mu$  take values over  $M$ , and use  $e$  as the 0 used in the definition of a concentrated set function above. It is then the addition operator  $\oplus$  which is to be used in the definition of the subadditive (or  $\mathcal{H}$ -subadditive) property and in the proof of the theorem. Thus, just by reading the proof of Dol'nikov under this new light we have:

**Theorem 2.** *Let  $\mu_i$ ,  $i = 1, \dots, k$  be  $k$  set functions defined on  $\mathcal{G}$  and taking values in a totally ordered unital magma  $(M, \oplus, \leq, e)$ . Let  $\delta_i \in M$  be such that  $(d - k + 2)\delta_i \leq \mu_i(\mathbb{R}^d)$ . If the functions  $\mu_i$  are monotone  $\mathcal{H}$ -subadditive and concentrated on bounded sets, there exists a  $(k - 1)$ -flat  $\pi$  such that*

$$\mu_i(h) \geq \delta_i, i = 1, \dots, k,$$

for all open halfspace  $h \in \mathcal{H}$  containing  $\pi$ .

### 3 Center transversal theorem for arrangements

Let  $A$  be an arrangement of  $n$  hyperplanes in the  $\mathbb{R}^d$ . We write  $V(A)$  for the set of all vertices (intersection points between any  $d$  hyperplanes) of  $A$  and  $CH(A) = CH(V(A))$  for the convex hull of those points. In the arguments that follow, by abuse of language, we will write  $A$  and mean  $V(A)$  or  $CH(A)$ . For example, we say that the arrangement  $A$  is *above hyperplane*  $h$  when all points in  $V(A)$  are above  $h$ . More generally for a region  $Q$  in the  $\mathbb{R}^d$ , we say that the arrangement  $A$  does not intersect  $Q$  if  $CH(A)$  does not intersect  $Q$ . We say that the  $k$  arrangements  $A_1, A_2, \dots, A_k$  are *disjoint* if their convex hulls do not intersect. They are *separable* if they are disjoint and no hyperplane intersects  $d + 1$  of them simultaneously.

Let  $\mathcal{H}$  be the set of all open halfspaces in  $\mathbb{R}^d$  and let  $\mathcal{G}$  be an additive family of subsets of  $\mathbb{R}^d$  that contains  $\mathcal{H}$ . For any set  $S \in \mathcal{G}$ , let  $\mu_A(S)$  be the maximum number of hyperplanes of  $A$  that have all their vertices inside of  $S$ , that is,

$$\mu_A(S) = \max_{A' \subseteq A, V(A') \subseteq S} |A'|.$$

In particular,  $\mu_A(\mathbb{R}^d) = \mu_A(CH(A)) = n$  and  $\mu_A(\emptyset) = d - 1$ .

#### 3.1 Lines in $\mathbb{R}^2$

We start with the planar case. Thus,  $A$  is a set of lines in  $\mathbb{R}^2$ , and  $\mathcal{H}$  is the set of all open halfplanes. Recall the Erdős-Szekeres theorem [ES35]

**Theorem 3** (Erdős-Szekeres). *For all integers  $r, s$ , any sequence of  $n > (r - 1)(s - 1)$  numbers contains either a non-increasing subsequence of length  $r$  or an increasing subsequence of length  $s$ .*

We show:

**Lemma 4.** *For any two sets  $S_1 \in \mathcal{H}$  and  $S_2 \in \mathcal{G}$ ,*

$$\mu_A(S_1 \cup S_2) \leq \mu_A(S_1)\mu_A(S_2)$$

*Proof.* Let  $\ell$  be a line defining two open halfplanes  $\ell^+$  and  $\ell^-$  such that  $S_1 = \ell^-$  and let  $S'_2 = S_2 \setminus \ell^-$ . Rotate and translate the plane so that  $\ell$  is the (vertical)  $y$  axis, and  $\ell^+$  contains all points with positive  $x$  coordinate. Let  $A'$  be a maximum cardinality subset of  $A$  such that  $V(A) \subseteq S_1 \cup S_2$ . Let  $l_1, \dots, l_{|A'|}$  be the lines in  $A'$  ordered by increasing order of their slopes, and let  $Y = (y_1, \dots, y_{|A'|})$  be the  $y$  coordinates of the intersections of the lines  $l_i$  with line  $\ell$ , in the same order. For any set  $A_1 \subseteq A'$  such that the  $y_i$  values of the lines in  $A_1$  form an increasing subsequence in  $Y$ , notice that  $V(A_1) \subseteq S_1$ . Likewise, for any set  $A_2 \subseteq A'$  that forms a non-increasing subsequence in  $Y$ , we have  $V(A_2) \subseteq S'_2$ . Any such set  $A_1$  is of size  $|A_1| \leq \mu_A(S_1)$  and any such set  $A_2$  is of size  $|A_2| \leq \mu_A(S'_2) \leq \mu_A(S_2)$ .

Therefore,  $Y$  has no non-decreasing subsequence of length  $\mu_A(S_1) + 1$  and no non-increasing subsequence of length  $\mu_A(S_2) + 1$ , and so by Theorem 3,  $\mu_A(S_1 \cup S_2) = |A'| = |Y| \leq \mu_A(S_1)\mu_A(S_2)$ .  $\square$

**Corollary 5.** *The set function  $\mu_A$  takes values in the totally ordered unital magma  $(\mathbb{R}_0^+, \cdot, \leq, 1)$ , it is monotone and  $\mathcal{H}$ -subadditive.*

We can thus apply the generalized center transversal theorem with  $k = 2$  to obtain a ham-sandwich cut theorem:

**Theorem 6.** *For any arrangements  $A_1$  and  $A_2$  of lines in  $\mathbb{R}^2$ , there exists a line  $\ell$  bounding closed halfplanes  $\ell^+$  and  $\ell^-$  and sets  $A_i^\sigma$ ,  $i \in 1, 2$ ,  $\sigma \in +, -$  such that  $A_i^\sigma \subseteq A_i$ ,  $|A_i^\sigma| \geq |A_i|^{1/2}$ , and  $V(A_i^\sigma) \in \ell^\sigma$ .*

Note that the bound in the above theorem is tight: assume  $n$  is the square of an integer. Construct the first line arrangement  $A_1$  with  $\sqrt{n}$  pencils of  $\sqrt{n}$  lines each, centered at points with coordinates  $(-1/2, i)$  for  $i = 1, \dots, \sqrt{n}$ , and the slopes of the lines in pencil  $i$  are distinct values in  $[1/2 - (i - 1)/\sqrt{n}, 1/2 - i/\sqrt{n}]$ . Thus all intersections other than the pencil centers have  $x$  coordinates greater than  $1/2$ . The line  $x = 0$  delimits two halfplanes in which  $\mu_{A_1}(x \leq 0) = \sqrt{n}$  since any set of more than  $\sqrt{n}$  lines have lines from different pencils which intersect on the right of  $x = 0$ , and  $\mu_{A_1}(x \geq 0) = \sqrt{n}$  since any set of more than  $\sqrt{n}$  lines has two lines in the same pencil which intersect left of  $x = 0$ . Since  $\mu_{A_1}$  is monotone, no vertical line can improve this bound on both sides. Perturb the points so that no two points have the same  $x$  coordinate. For  $A_2$ , build a copy of  $A_1$  translated down, far enough so that no line through two vertices of  $A_1$  intersects  $CH(A_2)$  and conversely. Therefore any line not combinatorially equivalent to a vertical line (with respect to the vertices of  $A_1$  and  $A_2$ ) does not intersect one of the arrangements and so there is no better cut than  $x = 0$ .

Applying the generalized center transversal theorem with  $k = 1$  gives a centerpoint theorem with a bound of  $|A|^{1/3}$ . A slightly more careful analysis improves that bound.

**Theorem 7.** *For any arrangement  $A$  of lines in  $\mathbb{R}^2$ , there exists a point  $q$  such that for every halfplane  $h$  containing  $q$  there is a set  $A' \subseteq A$ ,  $|A'| \geq \sqrt{|A|/3}$ , such that  $V(A') \in h$ .*

*Proof.* Let  $H$  be the set of halfplanes  $h$  such that  $\mu_A(h) < z = \sqrt{|A|/3}$ . The halfspace depth  $\delta(q)$  is the minimum value of  $\mu_A(h)$  for any halfspace containing  $q$ . Therefore, the region of depth  $\geq z$  is the intersection of the complements  $\bar{h}$  of the halfplanes  $h \in H$ . If there is no point of depth  $\geq z$  then the intersection of the complements of halfplanes in  $H$  is empty, and so (by Helly's Theorem) there must be 3 halfplanes  $h_1, h_2$ , and  $h_3$  in  $H$  such that the intersection of their complements  $\bar{h}_1 \cap \bar{h}_2 \cap \bar{h}_3$  is empty. But then, there is at least one point  $q \in h_1 \cap h_2 \cap h_3$ . Let  $h'_i$  be the translated halfplanes  $h_i$  with point  $q$  on the boundary. Since  $h'_i \subseteq h_i$ ,  $\mu_A(h'_i) \leq \mu_A(h_i) < z$ . The point  $q$  and the 3 halfplanes through it are witness that there is no point of depth  $\geq z$ .

The 3 lines bounding those 3 halfplanes divide the plane into 6 regions. Every line misses one of the three regions  $h'_1 \cap \overline{h'_2} \cap \overline{h'_3}$ ,  $\overline{h'_1} \cap h'_2 \cap \overline{h'_3}$ , and  $\overline{h'_1} \cap \overline{h'_2} \cap h'_3$ . Classify the lines in  $A$  depending on the first region it misses, clockwise. The largest class  $A'$  contains  $\geq |A|/3$  lines. Assume without loss of generality that all lines in  $A'$  miss  $h'_1 \cap \overline{h'_2} \cap \overline{h'_3}$ , then all intersections between lines of  $A'$  are in  $h'_2 \cup h'_3$ . By Lemma 4,

$$|A|/3 \leq |A'| = \mu_{A'}(h'_2 \cup h'_3) \leq \mu_{A'}(h'_2)\mu_{A'}(h'_3) < z^2 = |A|/3,$$

a contradiction. □

### 3.2 Hyperplanes in $\mathbb{R}^d$

We first briefly review a bichromatic version of Ramsey's theorem for hypergraphs.

**Theorem 8.** *For all  $p, a, b \in \mathbb{N}$ , there is a natural number  $R = R_p(a, b)$  such that for any set  $S$  of size  $R$  and any 2-colouring  $c : \binom{S}{p} \rightarrow \{1, 2\}$  of all subsets of  $S$  of size  $p$ , there is either a set  $A$  of size  $a$  such that all  $p$ -tuples in  $\binom{A}{p}$  have colour 1 or a set  $B$  of size  $b$  such that all  $p$ -tuples in  $\binom{B}{p}$  have colour 2.*

**Lemma 9.** *For any two sets  $S_1 \in \mathcal{H}$  and  $S_2 \in \mathcal{G}$ ,*

$$\mu_A(S_1 \cup S_2) \leq R_d(\mu_A(S_1) + 1, \mu_A(S_2) + 1) - 1$$

*Proof.* Let  $h$  be a hyperplane defining two open halfplanes  $h^+$  and  $h^-$  such that  $S_1 = h^-$  and let  $S'_2 = S_2 \setminus h^-$ . Let  $A'$  be a maximum cardinality subset of  $A$  such that  $V(A) \subseteq S_1 \cup S_2$ . Colour every subset of  $d$  hyperplanes in  $A'$  with colour 1 if their intersection point is in  $h^-$  and with colour 2 otherwise.

For any set  $A_1 \subseteq A'$  such that all subsets in  $\binom{A_1}{d}$  have colour 1, notice that  $V(A_1) \subseteq S_1$ . Likewise, for any set  $A_2 \subseteq A'$  such that all subsets in  $\binom{A_2}{d}$  have colour 2, we have  $V(A_2) \subseteq S'_2$ . Any such set  $A_1$  is of size  $|A_1| \leq \mu_A(S_1)$  and any such set  $A_2$  is of size  $|A_2| \leq \mu_A(S'_2) \leq \mu_A(S_2)$ .

Therefore,  $A'$  has no subset of size  $\mu_A(S_1) + 1$  that has all  $d$ -tuples of colour 1, and no subset of size  $\mu_A(S_2) + 1$  that has all  $d$ -tuples of colour 2, and so by Ramsey's Theorem,  $\mu_A(S_1 \cup S_2) = |A'| \leq R_d(\mu_A(S_1) + 1, \mu_A(S_2) + 1) - 1$ . □

Define the operator  $\oplus$  as  $a \oplus b = R_d(a + 1, b + 1) - 1$ . The operator is increasing and closed on the set  $\mathbb{N}_{\geq d-1}$  of naturals  $\geq d - 1$ . Since  $R_d(d, x) = x$  for all  $x$ ,  $d - 1$  is a neutral element. Therefore  $(\mathbb{N}_{\geq d-1}, \oplus, \leq, d - 1)$  is a totally ordered unital magma. Thus we have:

**Corollary 10.** *The set function  $\mu_A$  takes values in the totally ordered unital magma  $(\mathbb{N}_{\geq d-1}, \oplus, \leq, d - 1)$ , it is monotone and  $\mathcal{H}$ -subadditive.*

Define  $Q(x, c) := \max\{y \in \mathbb{N} \mid \oplus^c y \leq x\}$ . Apply now the generalized center transversal theorem to obtain:

**Theorem 11.** *Let  $A_1, \dots, A_k$  be  $k$  sets of hyperplanes in  $\mathbb{R}^d$ . There exists a  $(k - 1)$ -flat  $\pi$  such that for every open halfspace  $h$  that contains  $\pi$ ,*

$$\mu_{A_i}(h) \geq Q(|A_i|, d - k + 2).$$

The special case when  $k = d$  gives a ham-sandwich cut theorem.

**Corollary 12.** *Let  $A_1, \dots, A_d$  be  $d$  sets of hyperplanes in  $\mathbb{R}^d$ . There exists a hyperplane  $\pi$  bounding the two closed halfspaces  $\pi^+$  and  $\pi^-$  and sets  $A_i^\sigma \subseteq A_i$ ,  $\sigma \in \{+, -\}$ , such that  $V(A_i^\sigma) \subseteq \pi^\sigma$  and  $|A_i^\sigma| \oplus |A_i^\sigma| \geq |A_i|$ .*

If the arrangement  $A$  has the property that no  $r + 1$  hyperplanes intersect in a common point,  $\mu_A(\pi) \leq r$  for any hyperplane  $\pi$ , and so by Lemma 9, if  $h$  is an open halfspace bounded by  $\pi$  and  $\bar{h} = \pi \cup h$  is the corresponding closed halfspace,  $\mu_A(\bar{h}) \leq \mu_A(h) \oplus r$ .

**Corollary 13.** *Let  $A_1, \dots, A_d$  be  $d$  sets of hyperplanes in  $\mathbb{R}^d$ , no  $r + 1$  of which intersect in a common point. There exists a hyperplane  $\pi$  bounding the two open halfspaces  $\pi^+$  and  $\pi^-$  and sets  $A_i^\sigma \subseteq A_i$ ,  $\sigma \in \{+, -\}$ , such that  $V(A_i^\sigma) \subseteq \pi^\sigma$  and  $(|A_i^\sigma| \oplus |A_i^\sigma|) \oplus r \geq |A_i|$ .*

## 4 Same-type lemma for arrangements

Center transversal theorems, and especially the ham-sandwich cut theorem, are a basic tools for proving many facts in discrete geometry. We show here how the same facts can be shown for hyperplane arrangements.

A transversal of a collection of sets  $X_1, \dots, X_m$  is a  $m$ -tuple  $(x_1, \dots, x_m)$  where  $x_i \in X_i$ . A collection of sets  $X_1, \dots, X_m$  has *same-type transversals* if all of its transversals have the same order-type.

Note that  $m \geq d + 1$  sets have same-type transversals if and only if any  $d + 1$  of them have same-type transversals. There are several equivalent definitions for these notions.

1. The sets  $X_1, \dots, X_{d+1}$  have same-type transversals if and only if they are *well separated*, that is, if and only if for all disjoint sets of indices  $I, J \subseteq \{1, \dots, d + 1\}$ , there is a hyperplane separating the sets  $X_i, i \in I$  from the sets  $X_j, j \in J$ .
2. Connected sets  $C_1, \dots, C_{d+1}$  have same-type transversals if and only if there is no hyperplane intersecting simultaneously all  $C_i$ . Sets  $X_1, \dots, X_{d+1}$  have same-type transversals if and only if there is no hyperplane intersecting simultaneously all their convex hulls  $C_i = CH(X_i)$ .

The same-type lemma for point sets states that there is a constant  $c = c(m, d)$  such that for any collection  $S_1, \dots, S_m$  of finite point sets in  $\mathbb{R}^d$ , there are sets  $S'_i \subseteq S_i$  such that  $|S'_i| \geq c|S_i|$  and the sets  $S'_1, \dots, S'_m$  have same-type transversals. We here show a similar result for hyperplane arrangements.

A function  $f$  is *growing* if for any value  $y_0$  there is a  $x_0$  such that  $f(x) \geq y_0$  for any  $x \geq x_0$ .

**Lemma 14.** *For any integers  $d, m$ , and  $r$ , there is a growing function  $f = f_{m,d,r}$  such that for any collection of  $m$  hyperplane arrangements  $A_1, \dots, A_m$ , in  $\mathbb{R}^d$ , where no  $r + 1$  hyperplanes intersect at a common point, there are sets  $A'_i \subseteq A_i$  such that  $|A'_i| \geq f(|A_i|)$  and the sets  $CH(A'_1), \dots, CH(A'_m)$  have same-type transversals.*

*Proof.* The proof will follow closely the structure of Matousek [Mat02]. First notice that the composition of two growing functions is a growing function. The proof will show how to choose successive (nested) subsets of each set  $A_i$ ,  $c$  times where  $c = c(m, d)$  only depends on  $m$  and  $d$  and where the size of each subset is some growing function of the previous one.

Also, it will suffice to prove the theorem for  $m = d + 1$ , and then apply it repeatedly for each  $d + 1$  tuple of sets. The resulting function  $f_{m,d,r}$  will be  $f_{d+1,d,r}^{(m)}$ , the repeated composition of  $f_{d+1,d,r}$ ,  $\binom{m}{d+1}$  times.

So, given  $d + 1$  sets  $A_1, \dots, A_{d+1}$  of hyperplanes in  $\mathbb{R}^d$ , suppose that there is an index set  $I \subseteq \{1, \dots, d + 1\}$  such that  $\cup_{i \in I} \text{CH}(A_i)$  and  $\cup_{i \notin I} \text{CH}(A_i)$  are not separable by a hyperplane and assume without loss of generality that  $d + 1 \in I$ . Let  $\pi$  be the ham sandwich cut hyperplane for arrangements  $A_1, \dots, A_d$  obtained by applying Corollary 13. Then, the two open halfspaces  $\pi^\sigma$ ,  $\sigma \in \{+, -\}$  bounded by  $\pi$  contain each and for each  $i \in [1, d]$  a subset  $A_i^\sigma \subseteq A_i$  such that  $V(A_i^\sigma) \subseteq \pi^\sigma$  and  $(|A_i^\sigma| \oplus |A_i^\sigma|) \oplus r \geq |A_i|$ . Furthermore, because  $\mu_{A_{d+1}}(\pi) \leq r$  and by Lemma 9,

$$\mu_{A_{d+1}}(\pi^+) \oplus \mu_{A_{d+1}}(\pi^-) \oplus r \geq \mu_{A_{d+1}}(\mathbb{R}^d) = |A_{d+1}|.$$

Assume without loss of generality  $\mu_{A_{d+1}}(\pi^+) \geq \mu_{A_{d+1}}(\pi^-)$ . Then  $\mu_{A_{d+1}}(\pi^+) \oplus \mu_{A_{d+1}}(\pi^+) \oplus r \geq |A_{d+1}|$ . For each  $i \in I$ , let  $A'_i = A_i^+$  and for each  $i \notin I$ , let  $A'_i = A_i^-$ . Let  $g(x) = \min\{y | y \oplus y \oplus r \geq x\}$ . Then  $g$  is a growing function, and  $|A'_i| \geq |A_i|$ .

At worse, we have to shrink the sets for each possible  $I$ ,  $2^d$  times. Therefore for  $m = d + 1$ , the function  $f$  in the statement of the theorem is a composition of  $g$ ,  $2^d$  times, and is a growing function.  $\square$

In the plane, the same-type lemma readily gives a positive portion Erdős-Szekeres Theorem. Recall that the Erdős-Szekeres (happy ending) theorem [ES35] states that for any  $k$  there is a number  $ES(k)$  such that any set of  $ES(k)$  points in general position in  $\mathbb{R}^2$  contains a subset of size  $k$  which is in convex position.

**Theorem 15.** *For every integers  $k, r$ , and  $c$ , there is an integer  $N$  such that any arrangement  $A$  of  $N$  lines, such that no  $r+1$  lines go through a common point, contains disjoint subsets  $A_1, \dots, A_k$  with  $|A_i| \geq c$  and such that every transversal of  $\text{CH}(A_1), \dots, \text{CH}(A_k)$  is in convex position.*

*Proof.* Let  $m = ES(k)$  and let  $f = f_{m,2,r}$  be as in Lemma 14. Let  $N$  be such that  $f(\lfloor N/m \rfloor) \geq c$ . Partition the set  $A$  of  $N$  lines into  $m$  sets  $A_1, \dots, A_m$  of  $N/m$  lines arbitrarily. Apply Lemma 14 to obtain sets  $A'_1, \dots, A'_m$  each of size at least  $c$ . Finally, choose one transversal  $(x_1, \dots, x_m)$  from the sets  $\text{CH}(A'_i)$  and apply the Erdős-Szekeres theorem to obtain a subset  $x_{i_1}, \dots, x_{i_k}$  of points in general position. Because the sets  $\text{CH}(A'_i)$  have the same type property, every transversal of  $\text{CH}(A'_{i_1}), \dots, \text{CH}(A'_{i_k})$  is in convex position.  $\square$

## 5 Graph Drawing

Formally, a vertex labelling of a graph  $G = (V, E)$  is bijection  $\pi : V \rightarrow [n]$ . A set of  $n$  lines in the plane labelled from 1 to  $n$  *supports*  $G$  with *vertex labelling*  $\pi$  if there exists a straight-line crossing-free drawing of  $G$  where for each  $i \in [n]$ , vertex labelled  $i$  in  $G$  is mapped to a point on a line  $i$ . A set  $L$  of  $n$  lines labelled from 1 to  $n$  *supports* an  $n$ -vertex graph  $G$  if for every vertex labelling  $\pi$  of  $G$ ,  $L$  supports  $G$  with vertex labelling  $\pi$ . In this context clearly it only makes sense to talk about planar graphs. We are interested in an existence of an  $n$ -vertex line set that supports all  $n$ -vertex planar graphs, that is, in an existence of a *universal* set of lines for planar graphs.

**Theorem 16.** *For some absolute constant  $c'$  and every  $n \geq c'$ , there exists no set of  $n$  lines in the plane that support all  $n$ -vertex planar graphs.*

The following known result will be used in the proof of this theorem.

**Lemma 17.** [DEK<sup>+</sup>10] *There is a planar triangulation on 6 vertices, denoted by  $G_6$ , with vertex labelling  $\pi$  such that the following holds for every set  $L$  of 6 lines labelled from 1 to 6, no two of which are parallel. For every straight-line crossing-free drawing,  $D$ , of  $G_6$  where for each  $i \in [n]$ , vertex labelled  $i$  in  $\pi$  is mapped to a point on a line  $i$  in  $L$ , there is a point that is in an interior face of  $D$  and in  $\text{CH}(L)$ .*

*Proof of Theorem 16.* Let  $L$  be any set of  $n \geq c' = 5N$  lines, where  $N$  is obtained from Theorem 15 with values  $k = 6$ ,  $c = 6$ ,  $r = 17$ .

[EBFK09] proved that for every  $n \geq 6$ , no set of  $n$  parallel lines supports all  $n$ -vertex planar graphs. Thus if  $L$  has at least 6 lines that are pairwise parallel, then  $L$  cannot support all  $n$ -vertex planar graphs.

[DEK<sup>+</sup>10] proved that for every  $n \geq 18$ , no set of  $n$  lines that all go through a common point supports all  $n$ -vertex planar graphs. Thus if  $L$  has at least 18 such lines, then  $L$  cannot support all  $n$ -vertex planar graphs.

Thus assume that  $L$  has no 6 pairwise parallel lines and no 18 lines that intersect in one common point. Then  $L$  has a subset  $L'$  of  $c'/5 \geq N$  lines no two of which are parallel and no 18 of which go through one common point. Then Theorem 15 implies that we can find in  $L'$  six sets  $A_1, \dots, A_6$  of six lines each, such that the set  $\{\text{CH}(A_1), \dots, \text{CH}(A_6)\}$  is in convex position. Assume  $\text{CH}(A_1), \dots, \text{CH}(A_6)$  appear in that order around their common “convex hull”.

Consider an  $n$ -vertex graph  $H$  whose subgraph  $G$  is illustrated in Figure 1.  $G \setminus v$  has three components,  $A$ ,  $B$ , and  $C$ , each of which is a triangulation. Each of the components  $A$ ,  $B$ , and  $C$  has two vertex disjoint copies of  $G_6$  (the 6-vertex triangulation from Lemma 17). Map the vertices of the first copy of  $G_6$  in  $A$  to  $A_1$  and the second copy to  $A_4$  using the mapping equivalent to  $\pi$  in Lemma 17. Map the vertices of the first copy of  $G_6$  in  $B$  to  $A_2$  and the second copy to  $A_5$  using the mapping equivalent to  $\pi$  in Lemma 17. Map the vertices of the first copy of  $G_6$  in  $C$  to  $A_3$  and the second copy to  $A_6$  using the mapping equivalent to  $\pi$  in Lemma 17. Map the remaining vertices of  $H$  arbitrarily to the remaining lines of  $L$ .

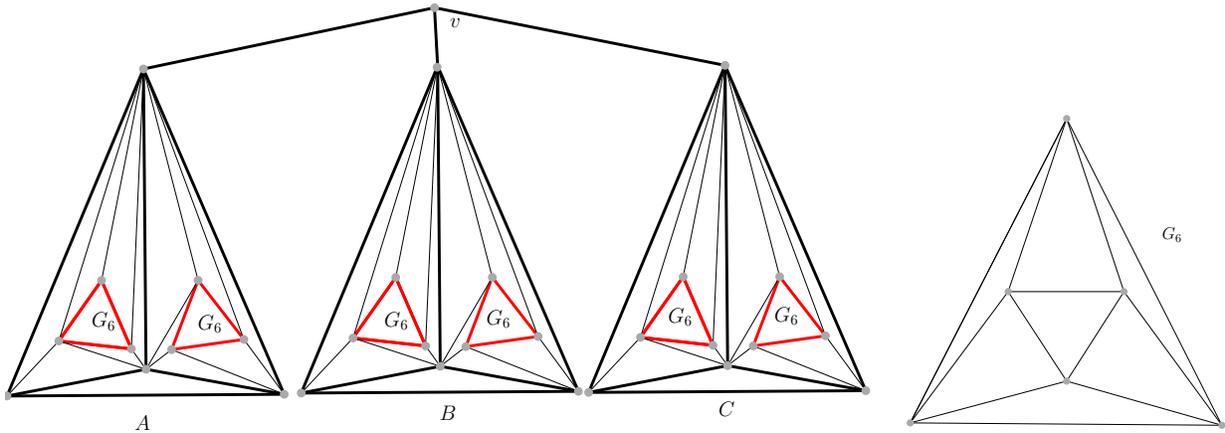


Figure 1: Illustration for the proof of Theorem 16.

Assume, for the sake of contradiction, there  $L$  supports  $H$  with such a mapping and consider the resulting crossing-free drawing  $D$  of  $H$ . In  $D$  the drawing of each of  $A$ ,  $B$ , and  $C$  has a triangle as an outerface. Let  $T_A$ ,  $T_B$ , and  $T_C$  denote these three triangles together with their interiors in the plane.

It is simple to verify that in any crossing-free drawing of  $G$  at least two of these triangles are disjoint, meaning that there is no point  $p$  in the plane such that  $p$  is in both of these triangles. Assume, without loss of generality, that  $T_A$  and  $T_B$  are disjoint. By Lemma 17, there is a point  $p_1 \in \text{CH}(A_1)$  such that  $p_1 \in T_A$ , and a point  $p_4 \in \text{CH}(A_4)$  such that  $p_4 \in T_A$ . Thus the segment  $\overline{p_1 p_4}$  is in  $T_A$ . Similarly, by Lemma 17, there is a point  $p_2 \in \text{CH}(A_2)$  such that  $p_2 \in T_B$ , and a point  $p_5 \in \text{CH}(A_5)$  such that  $p_5 \in T_B$ . Thus the segment  $\overline{p_2 p_5}$  is in  $T_B$ .

By Theorem 15,  $\overline{p_1 p_4}$  and  $\overline{p_2 p_5}$  intersect in some point  $p$ . That implies that  $p \in T_A$  and  $p \in T_B$ .

That provides the desired contradiction, since  $T_A$  and  $T_B$  are disjoint.  $\square$

As a sharp contrast to Theorem 16, the following theorem shows that the situation is starkly different for unlabelled planar graphs. Namely, every set of  $n$  lines supports all  $n$ -vertex unlabelled planar graphs. The proof of this theorem does not use any of the tools we introduced in the previous section and is in that sense elementary. But to our knowledge, it has not appeared in the literature before.

**Theorem 18.** *Given a set  $L$  of  $n$  lines in the plane, every planar graph has a straight line crossing free drawing where each vertex of  $G$  is placed on the distinct line of  $L$ . (In other words, given any set  $L$  of lines, labelled from 1 to  $n$ , and any  $n$ -vertex planar graph  $G$  there is a vertex labelling  $\pi$  of  $G$  such that  $L$  supports  $G$  with vertex labelling  $\pi$ .)*

*Proof.* In this proof we will use canonical orderings introduced in [dFPP90] and a related tool called frame introduced in [BDH<sup>+</sup>09]. We first recall these tools. We can assume  $G$  is an embedded edge maximal planar graph.<sup>2</sup> Each face of  $G$  is bounded by a 3-cycle. De Fraysseix [dFPP90] proved that  $G$  has a vertex ordering  $\sigma = (v_1, v_2, v_3, \dots, v_n)$ , called a *canonical ordering*, with the following properties. Define  $G_i$  to be the embedded subgraph of  $G$  induced by  $\{v_1, v_2, \dots, v_i\}$ . Let  $C_i$  be the subgraph of  $G$  induced by the edges on the boundary of the outer face of  $G_i$ . Then

- $v_1, v_2$  and  $v_n$  are the vertices on the outer face of  $G$ .
- For each  $i \in \{3, 4, \dots, n\}$ ,  $C_i$  is a cycle containing  $v_1 v_2$ .
- For each  $i \in \{3, 4, \dots, n\}$ ,  $G_i$  is biconnected and *internally 3-connected*; that is, removing any two interior vertices of  $G_i$  does not disconnect it.
- For each  $i \in \{3, 4, \dots, n\}$ ,  $v_i$  is a vertex of  $C_i$  with at least two neighbours in  $C_{i-1}$ , and these neighbours are consecutive on  $C_{i-1}$ .

For example, the ordering in Figure 2(a) is a canonical ordering of the depicted embedded graph  $G$ .

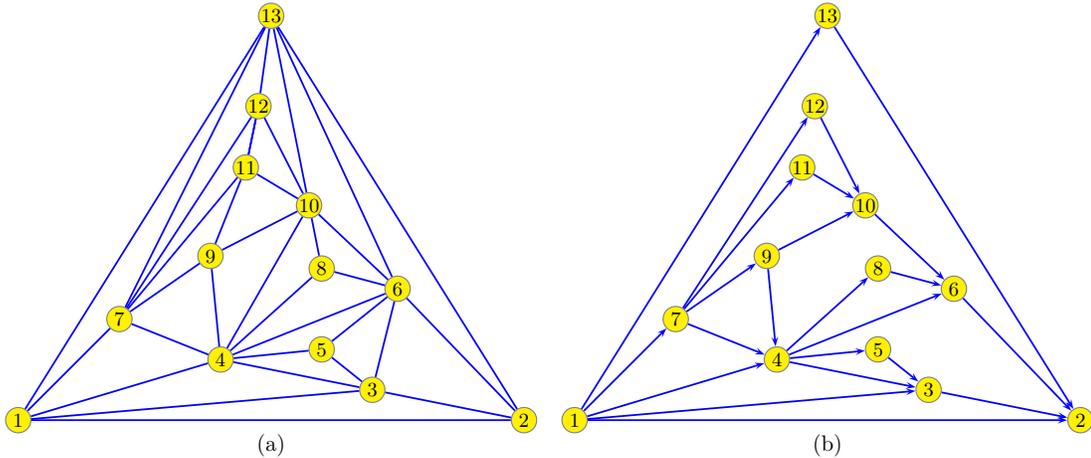


Figure 2: Illustration for the proof of Theorem 18: (a) Canonical ordering of  $G$ , (b) Frame  $\mathcal{F}$  of  $G$

A *frame*  $\mathcal{F}$  of  $G$  [BDH<sup>+</sup>09] is the oriented subgraph of  $G$  with vertex set  $V(\mathcal{F}) := V(G)$ , where:

<sup>2</sup>A planar graph  $H$  is edge-maximal (also called, a *triangulation*), if for all  $vw \notin E(H)$ , the graph resulting from adding  $vw$  to  $H$  is not planar.

- $v_1v_2$  is in  $E(\mathcal{F})$  and is oriented from  $v_1$  to  $v_2$ .
- For each  $i \in \{3, 4, \dots, n\}$  in the canonical ordering  $\sigma$  of  $G$ , edges  $pv_i$  and  $v_ip'$  are in  $E(\mathcal{F})$ , where  $p$  and  $p'$  are the first and the last neighbour, respectively, of  $v_i$  along the path in  $C_{i-1}$  from  $v_1$  to  $v_2$  not containing edge  $v_1v_2$ . Edge  $pv_i$  is oriented from  $p$  to  $v_i$ , and edge  $v_ip'$  is oriented from  $v_i$  to  $p'$ , as illustrated in Figure 2(b).

By definition,  $\mathcal{F}$  is a directed acyclic graph with one source  $v_1$  and one sink  $v_2$ . The frame  $\mathcal{F}$  defines a partial order  $<_{\mathcal{F}}$  on  $V(\mathcal{F})$ , where  $v <_{\mathcal{F}} w$  whenever there is a directed path from  $v$  to  $w$  in  $\mathcal{F}$ .

Translate the given set  $L$  of lines so that all vertices of the arrangement of lines have negative  $y$  coordinates, and sort the lines  $\ell_i \in L$  according to the  $x$  coordinate  $b_i$  of the intersection of  $\ell_i$  with the  $x$  axis. Therefore, the lines  $\ell_i \in L$  have equation  $y = a_i(x - b_i)$ , with  $b_1 < b_2 < \dots < b_n$ . Because all intersections among lines of  $L$  have negative coordinates, all  $b_i$  are distinct, and the values  $1/a_i$  are sorted. Note that the slopes  $a_i$  might be positive or negative. Let  $\hat{a} = \min |a_i|$ . For any segment of slope in  $[-\hat{a}, \hat{a}]$  connecting two points  $(x_i, y_i) \in \ell_i$  and  $(x_j, y_j) \in \ell_j$  above the  $x$  axis (that is,  $y_i, y_j > 0$ ),  $x_i < x_j$  if and only if  $i < j$ .

Construct a linear extension  $v_1, v_2, \dots, v_n$  of the partial order  $<_{\mathcal{F}}$  and define the bijection  $\pi : V \rightarrow [n]$  as  $\pi(v_i) = i$ . That is, the vertices of  $G$  will be placed on the lines in such a way that the partial order  $<_{\mathcal{F}}$  is compatible with the order determined by the values  $b_i$  of the lines.

We prove by induction that for every value  $\hat{y}$  and every  $i \geq 2$ , it is possible to draw  $G_i$  such that  $v_1$  and  $v_2$  are placed on points  $(b_1, 0)$ ,  $(b_n, 0)$ , and the  $y$  coordinates of all other vertices are in the horizontal slab  $(0, \hat{y}]$ . The base case ( $i = 2$ ) is obviously true.

Note that we could have formulated the induction on the slopes of the edges of  $G_i$  in the drawing. In fact those two formulations imply each other: for any value  $0 < s \leq \hat{a}$ , there is a  $\hat{y}_s > 0$  such that any segment whose endpoints lie on distinct lines of  $L$  and have  $y$  coordinates in  $[0, \hat{y}_s]$ , the slope of the segment is in  $[-s, s]$ . This is easy to see: draw an upward cone with apex on each point  $(b_i, 0)$  and bounded by the lines of slopes  $s$  and  $-s$  through that point. Define  $\hat{y}_s$  as the  $y$  coordinate of the lowest intersection point between any two such cones. Any segment with a slope not in  $[-s, s]$  and with its lowest point inside a cone must have its highest point inside the same cone, therefore no segment connecting two different lines inside the horizontal slab  $[0, \hat{y}_s]$  can have such a slope.

Assume by induction that the statement is true for  $G_{i-1}$ . We will show how to draw  $G_i$  for a specific value  $\hat{y}$ . The point  $v_i$  will be placed on the point on line  $i$  with  $y$  coordinate  $\hat{y}$ . Let  $s_1$  and  $s_2$  be the slopes of the segments  $v_1v_i$  and  $v_iv_2$ , and let  $s = \max(|s_1|, |s_2|)/2$  or  $\hat{a}$ , whichever is smaller. Let  $y_1$  be the intersection of the line of slope  $s$  through  $v_i$  and line  $\ell_1$  and  $y_2$  the intersection of the line of slope  $-s$  through  $v_i$  and  $\ell_n$ . Note that  $y_1$  and  $y_2$  are strictly positive. Let  $\hat{y}' = \min(y_1, y_2, \hat{y}_s)$ . Apply the induction hypothesis to draw  $G_{i-1}$  in the horizontal slab  $[0, \hat{y}']$ . Thus, in the drawing of  $G_{i-1}$ , all edges have slope at most  $s \leq \hat{a}$ . Then by construction, the path in  $C_{i-1}$  from  $v_1$  to  $v_2$  not containing edge  $v_1v_2$  is  $x$ -monotone (that is, all its edges are oriented rightwards), and  $v_i$  is above the supporting line of each edge on that path. Therefore,  $v_i$  can see all vertices in  $C_{i-1}$  and all edges adjacent to  $v_i$  can be drawn.  $\square$

We conclude this part with an intriguing 3D variant of this graph drawing problem. A graph is *linkless* if it has an embedding in 3D such that no two cycles of the graph have nonzero linking number. These graphs form a three-dimensional analogue of the planar graphs. Is there an arrangement of labelled planes in 3D such that any labelled linkless graph has a linkless straight-line embedding where each vertex is placed on the plane with the same label?

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